

A Concrete Approach to Teaching Symbolic Algebra

Stephen Norton
Griffith University
<s.norton@griffith.edu.au>

Jane Irvin
Griffith University

Student difficulties with the study of algebra have been well documented. The inability of many students to understand variables and formal symbolic manipulation act as a barrier to success in mathematics study. This report documents an intervention that uses a concrete approach to teaching algebra in a Year 9 class. Results indicate that much of the student struggle was associated with a lack of understanding of arithmetic concepts including those associated with equivalence, operations with negative integers, and the distributive law and fraction concepts. Once these difficulties were addressed through the explicit teaching of the links between materials and symbols, materials and language, language and symbols, students made considerable progress in writing, simplifying expressions, and solving equations with variables on both sides.

Introduction and Background

This paper reports on a teacher's (Jane) attempts to teach critical algebra understandings, in particular, how to solve equations with variables on both sides. Jane is the mathematics subject head of department in a suburban high school situated in a middle to lower class outer Brisbane suburb. Historically very few students in the school opted to study intermediate or advanced mathematics and Jane hoped to increase the proportion of students enrolling these courses (Mathematics B and Mathematics C). To this end Jane devised an algebra intervention for Year 9, in which she hoped that student success in middle school algebra would encourage a higher proportion of the students to enrol in the more advanced senior mathematics subjects. This paper describes the intervention (in brief) and reports on the barriers to, and successes in, student learning of algebra when a verbal and concrete approach to teaching was undertaken. Stacey and Chick (2004) noted "The algebra teacher has a crucial role to play both in bringing algebraic representations to the fore and in making their manipulation by students a venue for epistemic growth" (p. 31).

Many students come to the study of early algebra with poor understandings of arithmetic (Thompson & Fleming, 2003). The use of calculators can account for some of the difficulties associated with number computation (MacGregor, 2004), however, it is likely that failure to understand the structures of arithmetic (e.g., commutative law, distributive law, fractions, integers and operations) will place an added cognitive load on students when it comes to the study of algebra. Kieran and Yerushalmy (2004, p. 21) described algebra as "Generalization of numerical and geometric patterns and the laws governing numerical relationships" and Sfard (1994) discussed algebra as "generalised arithmetic" consisting of the "operational" and "structural" phases. Sfard's (1994) definition of "operational algebra" can be summed up as being tied to arithmetic operations, for example, the use of backtracking to solve simple linear equations can be seen as the reversal of arithmetic operations. "Structural algebra" can be seen in solving an equation with variables on both sides, however, simple reversal of operations such as in backtracking does not suffice. The solution requires the suspension of operational thinking to view the overall structure of the equation, that is, "structural" thinking. Stacey and MacGregor (1999) regarded students' ability to solve equations with variables on both sides as an indicator of "formal algebra" or what Sfard (1994) regarded as "structural algebra". The ability of students to solve such equations can be seen as a marker between arithmetic and algebraic thinking.

Stacey and Chick (2004) noted an important part of algebra learning is transformational processes. Clearly, without the transformational tools of arithmetic, students are likely to be burdened with added cognitive load and struggle to move from operational to the structural phase of algebra thinking. Another way of putting this is to say that without a foundation of numeracy the “generalization” of it would seem to be a more difficult task, some would say an impossible task, unless the structures of arithmetic were made explicit and taught simultaneously with algebra, at least as far as can be done. In addition, Lins and Kaput (2004) support this position emphasising the parallels between fundamental processes of arithmetic and algebra.

Jane’s concerns about the proportion of students undertaking more advanced mathematics are shared by the broader mathematics community (e.g., Barrington, 2006). It has previously been reported that traditional school algebra is not appropriate for students with weak literacy and numeracy skills and that these students may prefer to acquire knowledge through increased verbal interaction and concrete activity, and that failure in early algebra is likely to lead to passive withdrawal from further study or active rebellion (MacGregor, 2004). In this way algebra study acts as a filter to the study of more advanced mathematics (e.g., MacGregor, 2004; Stacey & Chick, 2004). Similarly, Jane’s focus on equivalence, expressions, variables and solving with variables on both sides of the equal sign have been described as critical to algebra (e.g., Bazzini, Boero, & Garuti, 2001; Herscovics & Linchevski, 1994; MacGregor & Stacey, 1997; Stacey & Chick, 2004; Stacey & MacGregor, 1999). It is generally recognised that traditional approaches to teaching algebra have failed. Booker (1987) summed up the difficulties with problems associated with the introduction of symbolic values as being a result of changes in language and nuances with respect to operations when students attempt to move from operating arithmetically to algebraically. Kaput (1987) puts the issues more bluntly, pointing out the perceived meaninglessness of school mathematics in general, and algebra in particular, as being at the heart of the problem. Kaput (1995, p. 4) reported that most students see algebra as “little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test.” In short, algebra makes little sense to many children. Solutions to the problem of algebra failure are many and frequently interconnected, and include the following:

- Making explicit algebraic thinking inherent in arithmetic in children’s earlier learning (e.g., Lins & Kaput, 2004; Warren & Cooper, 2006).
- Explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (e.g., Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997, 1999; Stacey & Chick, 2004), especially in transformational activities (e.g., Kieran & Yerushalmy, 2004; Stacey & Chick, 2004).
- Using multiple representations including the use of technology (e.g., Kieran & Yerushalmy, 2004; Van de Walle, 2006).
- Recognising the importance of embedding algebra into contextual themes (National Council of Teachers of Mathematics, 1998; Stacey & Chick, 2004).

Clearly, much more can be said about the scope of algebra research, however, this is a brief paper. A review of the literature reveals that as more and more is written the terminology becomes increasing specialised, but the problems have persisted over 20 years of algebra teaching reform. One explanation is that top down reform recommendations have been difficult to implement in the classroom. In this study, the reforms reported have been generated from a teacher’s perceptions of student needs and implemented as a reform of pedagogy in her classroom.

Method

The overall design is a case study that uses design based research, in so much as cycles of design, enactment, analysis and re-enactment, analysis, and further design take place. As in all design-experiments, the specific research questions investigated in each iteration are conjured out of analysis of recent failures of previous iterations (Bereiter, 2002). This study reports on Jane's third iteration of the intervention in 2006, but each iteration was essentially identical in terms of teaching approach. This iteration was the beginning of the researcher's engagement with the school algebra project. Future iterations will reflect what has been learnt from the analysis reported in this paper. The involvement of the researcher as an active participant in this process gave the research design a participatory collaborative action research element (Kemmis & McTaggard, 2000).

Participants

The participants in this study were the classroom teacher, Jane, and the 18 students engaged in a 6 week algebra course. The school was a State School located in a middle to low socio-economic status suburb. In recent years between 4% and 7% of the senior school had enrolled in Mathematics C (Advanced Mathematics). Although approximately 25% enrolled in Mathematics B (Intermediate Mathematics), half of these students failed and or withdrew in Year 11, leaving approximately 12% entering Year 12. In comparison, the national average enrolment in 2004 for Advanced Mathematics was 11.7% and for Intermediate Mathematics it was 22.7% (Barrington, 2006). The students in the study were drawn from the 180 students in the Year 9 cohort. All 180 students were tested for general numeracy and more specifically to determine those who were "comfortable with the use of symbols to describe patterns" (Jane, personal communication, 2007). Students who scored in the top 1/3 on the pre-test were offered the algebra extension. There were three cohorts of about 20 students each. The intervention occurred in 18 one-hour lessons over 6 weeks.

Data collection and Analysis

All 18 lessons were observed and video recorded over the 6 weeks, including recording of class discussions, examples of student working on tasks in small groups, and examples of the teacher and researcher scaffolding student learning. Student work samples including workbooks, tests, and scripts were collected. Students were asked to explain their reasons for making mathematical decisions throughout the duration of the study. Student work was analysed for error patterns. In the case of their test scripts errors in computation and transformation could be seen in their recording of their mathematical processes. This also occurred in examining their class work. Additionally, in class students asked why they made mathematical decisions. Finally, the nature of student difficulties could be deduced from the questions they asked Jane and the discussions they had with their peers during group work.

Results and Discussion

Description of Instructional Discourse.

Instructional discourse refers to the rules for selecting and organising instructional content (Bernstein, 2000). Jane articulated her intentions as follows, "They needed to experience mathematics study in an academic and rigorous way." The instructional discourse was based on an underpinning theoretical framework put forward by Booker, Bond, Sparrow, and Swan (2004, p. 20).

While the role of materials and patterns they develop is fundamental, materials by themselves do not literally carry meaning...it is language that communicated ideas, not only in describing concepts but also helping them take shape in each learner's mind.

Jane's selection of activity sources was based on helping students make connections between materials, verbal language initially, and then symbolic language. The primary sources of activity were *A Concrete Approach to Algebra* (Quinlan, Low, Sawyer, White, & Llewellyn, 1987) and *Access to Algebra Book 2* (Lowe, Johnston, Kissane, & Willis, 1993). These resources used unmarked cups with hidden counters (blobs), envelopes with hidden counters to help develop the concept of variables, and extensive use of other concrete materials including patterns made from counters or match sticks. Both resources emphasised the use of language and logic to connect patterns modelled with material to verbal descriptions of the patterns, tabular summaries of the patterns and symbolic representations. Jane used match stick patterning to introduce variables and activities with cups, counters (blobs) and envelopes to explore writing expressions, equivalent expressions, simplifying expressions, expanding expressions and writing equivalent equations. Equations were created and solved using the balance model, initially with the concrete materials and, then, linking to traditional recording using symbols. Activities from Lowe et al. (1993), were selected that emphasised the links between materials and symbols. In this way students saw the meaning of the equals sign in the context of an algebraic equation. They also learnt the careful recording of transformations on both sides of the equation. The third source of student activities was based on the symbolic recognition and manipulations of algebra terms covered above embedded in algebra games that Jane had devised. The algebra games were constructed according to principles outlined by Booker (2000), some were track or strategy grid board games in which diagrammatic representations of concrete materials needed to be matched with symbolic expressions. Other games included concept games in which randomness of question was introduced by throwing dice of various configurations. For example, a concept game required players to write an algebraic equation from a scenario given in words and then solve the equation: A *number* is multiplied by Δ (a ten sided die is rolled to provide this number), then \odot is added to it (a second 10 sided die is rolled to provide this number), the answer is \bigcirc (a 36 sided die is rolled to provide this number), what is the *number*? Such an equation is linear with a variable on one side of the equals sign. It can be solved using the balance model and frequently results in a fraction solution. The games could be played by two or three students, and enabled them to consolidate and attain competency in the mathematics learnt in prior activities.

Description of Regulatory Discourse

Regulative discourse refers to the models of the teacher, learner and, pedagogic relations that underpin the selection and organisation of content within learning activities (Bernstein, 2000). Typically, the 1-hour lessons were divided into three segments. In an introductory segment, Jane used the white board and an activity selected from Quinlan et al. (1987) or Lowe et al. (2001) as the basis to conduct a class discussion on the key concepts. During the segment she kept a careful record of the discourse on the white board. In this discourse, Jane emphasised the links between materials, natural language which she extended to the nuances of algebraic language, and symbols. Typically, in the second segment, students worked in pairs or threes on activities selected from Quinlan et al. (1987) or Lowe et al. (1993) and Jane helped individuals or pairs of students when they requested assistance. Sometimes this activity continued to the end of the class. Generally, the third segment was used by students to play the algebra games designed to give students an opportunity to apply and consolidate the algebra learning that had occurred earlier.

Results of Discourse

The results of this discourse are presented in two sections. First, the types of errors that limited student completion of the algebra tasks are presented. Second, the success or otherwise of students on a written test and an analysis of their errors is presented.

Video analysis of teacher/student discussion indicated that the following difficulties and/or errors were most common in limiting student understanding and completion of the algebra based activities.

1. Difficulties associated with operations with negative integers (e.g., $4 - -3$; $-4 + -2$; $-3 - -7$). Students did not know how to complete these computations. In addition students experienced difficulties with subtraction signs when expanding, for example $2(4 - 5)$, students ignoring the $-$ sign and treating it as an addition obtaining an answer of 18; and $3(2x - 4)$ expanded to $6x + 12$.
2. Difficulties associated with solving equations of the form $3x + 3 = 15$. In particular, students not treating the equal sign as an indication that equivalence must be maintained. For example, students removed the 3 from the left hand side but not the right hand side, thus solving for x as equal to 5. Similar mistakes were made on equations such as $x - 2 = 2x + 3$ where students would add 2 to the LHS but not to the RHS. When students were first challenged with problems of this structure, some attempted to use “backtracking” and simply reported it could not be done.
3. Difficulties associated with number facts, such as students not knowing their multiplication facts and making computational errors.
4. Difficulties associated with fractions, such as errors in solving equations of the form $3y + 18 = 6y + 6$; students responding with $y + 18 = 2y + 6$ indicating that students had generalised inappropriately about cancelling. In this instance the error has its roots in arithmetic where students are taught to simplify fraction computations by cancelling. For example, in operating upon the fraction below (e.g., $(2 + 3)$ divided by 2), students simply cancelled the 2s and answered 3.

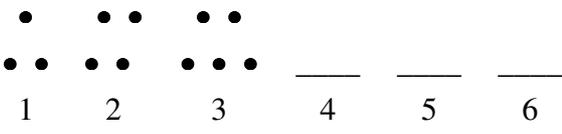
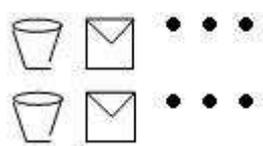
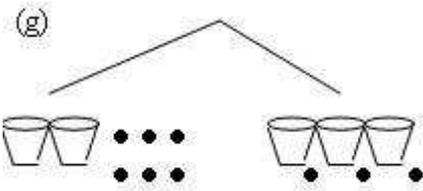
This over generalisation in regard to fraction cancelling results from an inadequate understanding of fractions, and the application of this limited understanding to the algebra solving problem above fails the student irrespective of the student’s understanding of symbolism. One of the goals of the teaching program was to address these difficulties within the teaching of the algebraic skills. Jane and the researcher’s approach when confronted with such problems in the context of algebra was to re-teach the concepts in arithmetic contexts (e.g., students adding $\frac{1}{2}$ to $\frac{3}{3}$ equal $\frac{4}{5}$); Jane would revise the concept of equivalence of fractions using paper fraction strips to display a visual model of equality or in equality, in this case one half is not equal to one third, before linking this to multiplication by unity (e.g., $\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$ to enable the formation of fractions with the same name or denominator). The approach of teaching arithmetic and algebra concurrently with the aid of concrete materials has found favour in those who recommend the teaching of algebra early in students study (e.g, Lins & Kaput, 2004; Warren & Cooper, 2006).

Summary of Written Test Results

A written post test consisting of 25 separate questions was completed by 15 students. One of the students missed many of the algebra lessons and her results were consistently incorrect. A sample of the questions and the number of students who answered them correctly are listed in Table 1. All students were able to recognise the pattern, complete the table of ordered pairs and represent it symbolically as equivalent to $p + 2 = n$. One student did not complete the equation. Seven of the students were able to correctly graph the function. Little class time was spent on graphing of variables. A number of authors have noted that multiple representations

of functions including the generation of tables and graphs assist student understanding of algebraic relationships (e.g., French, 2002; Kieran & Yerushalmy, 2004). French (2002, p. 81) commented that “students need to understand the links between the equation, the table of values or set of co-ordinates and the graph, and to be able to move fluently between these representation.” In this regard the use of technologies such as excel spread sheets and graphing calculators has been recommended (e.g, Kieran & Yerushalmy, 2004; Kissane, 1999). Clearly, this was an instructional discourse issue to be addressed in future algebra teaching in this school.

Table 1
Summary of Test Results for 15 Students

Concept	Typical question	Correct responses																		
Completing a pattern, table, and describing the pattern algebraically.		14/15 1 partial																		
	<table border="1" data-bbox="470 929 1117 1019"> <tr> <td>P</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> <td>20</td> <td></td> </tr> <tr> <td>N</td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td>102</td> </tr> </table>	P	1	2	3	4	5	6	20		N								102	
P	1	2	3	4	5	6	20													
N								102												
Writing expressions and equations	(f) 	14/15																		
	(g) 	13 /15 1 partial																		
Simplify expressions	$7x - 2x + 5y - 3y$	14/15																		
Expand and simplify	$3(3x - 2y)$ $b(x + 2y)$	14/15 8/15																		
Solving equations with model	$3y + 2 = y + 6$	10/15, 2 partial answer.																		
Solving equations without a model	$5x + 2 = 7x - 9$	5/15 correct, 2 partial correct.																		

Almost all students were able to write the symbolic expression given a pictorial representation. For example, all but two students could transform an equation represented with cups and counters into an algebraic equation (see question g; $2y + 6 = 3y + 3$). These findings suggest that student understanding of the variable concept was progressing, in that students used symbols to represent variables in an unknown context. These findings are in

contrast with those of MacGregor and Stacey (1997) who reported that the majority of students in a broad Australian study up to age 15 seemed unable to interpret algebraic letters as generalised numbers or even specific unknowns. MacGregor and Stacey found students ignored letters, replaced them with numerical numbers or regarded them as short hand names. For example, some students viewed letters in algebra as abbreviated words, whereas others the letter with its place in the alphabet (as occurs in some puzzles and code translations). In addition, MacGregor and Stacey noted that students writing of letters in contexts such as $h10$ meant add “10 to h” and $1y$ meant take one from y, indicative of the Roman subtraction principle. Clearly, some of these errors arise out of the inappropriate transfer of generalisations. Of additional concern to MacGregor and Stacey was the prevalence of students being unable to distinguish the name of the object (e.g., the person Con) from the name of the attribute (e.g., Con’s height). Such errors are a serious obstacle to writing expressions and equations. Such errors were not evident in the final written tests or during class in the latter stages of the intervention in this study. The findings that almost all the students could interpret and simplify the cups and counters equation representations correctly is encouraging and in contrast to the results reported by MacGregor and Stacey (1997). Essentially, this meant that the students recognised that x and y were symbolic representations of a variable (generally) and could complete simple arithmetic computations involving the symbols.

Almost all students expanded $3(3x - 2y)$ correctly, but less than half of these students were able to expand $b(x + 2y)$ appropriately. This suggests that the students might not have an understanding of multiplication separate from repeated addition. Subsequent to reviewing these results Jane reported that she had believed that the way she taught expansion by using concrete materials encouraged the students to use repeated addition at first. She had hoped for them to then establish a pattern which would mature to the full understanding of the distributive law. Jane said she was attempting to assist the students to develop a full understanding rather than a superficial procedural knowledge likely to be generated by the usual approach to expansion such as drawing arrows from the 3 to the $3x$ and $-2y$. The test scripts supported her preferred approach for treating $3(3x - 2y)$. However, those students who could not expand $b(x + 2y)$ expanded $3(3x - 2y)$ using the repeated addition algorithm as follows (Figure 1):

$3(2x - 2y) =$	$6x - 6y$
$2x - 2y$	
$+ 2x - 2y$	
$2x - 2y$	
$\hline 6x - 6y$	

Figure 1. Teaching expansion

When the variable in front is included, as in $b(x + 2y)$, the repeated addition model is no longer an available strategy. However, students with a good understanding of the distributive law, for example, being able to view 14×3 as $(10 + 4)$ multiplied by 3, which can be taught with a focus on place value (i.e., 4 ones multiplied by 3 ones is 12 ones, renamed as 2 ones and 1 ten; 1 ten multiplied by 3 ones is 3 tens, added the renamed ten gives a total of 4 tens and 2 ones or 42 ones), ought to have been able to make the transition. Most did not. When this early number teaching is linked to the array model and the application of the distributive law, the number multiplication $3(10 + 4)$ has exactly the same structure $b(10 + 4)$ and the similarity in structure can be extended to $b(x + 2y)$. This example illustrates the opportunity

to capitalise on an understanding of arithmetic structures in the learning of algebra. In this study the use of the array model in linking the application of the distributive law in number and algebra was not made explicit, hence it might reasonably be argued that the student results reflected this omission.

Almost all students solved an algebraic equation with unknowns on both sides using materials (Table 1 – Solving equation with model), and one third of the students solved a similar structured equation without the use of materials (Table 1 – Solving equation without model). It could be said that those students who completed the solving task without materials had developed an abstract schema of variables while those who solved the equation with materials but not without, were at an intermediate stage. Ability to equation solve such as that above has been described as achieving beyond a didactic cut or cognitive gap (Herscovics & Linchevski, 1994) and is a critical indicator of algebraic thinking. Similarly, Stacey and MacGregor (1999) regard this type of problem solving as an indicator of formal algebra capacity. This is the case since the equation cannot be easily solved arithmetically, algebraic competence is required (Stacey & MacGregor, 1999). Stacey and MacGregor reported that only about 8% of Year 10 students made this cut, those failing tending not to use logical reasoning in relation to inverse operations, instead using guess and check methods or attempting to use numerical methods; that is, they could be described as not reasoning algebraically.

Encouragingly, there was no evidence at the end of the study that students retained misconceptions about symbolism including confounding with place value, letters standing for abbreviations or for specific numbers, misuse of conventions (e.g., work from left to right), and false analogies with ordinary language such as that described by Stacey and MacGregor (1997) and Sleeman (1986).

Conclusions and Recommendations

The activities in this intervention were not applied or linked to authentic contexts or real world situations. This was almost pure algebra with a heavy focus upon the development of symbolic meaning and symbolic manipulation through the use of concrete materials. The results cause us to qualify the recommendations of the NCTM (1998) that the teaching of algebra be tied to contextual themes. The relative success of students in writing expressions and solving equations reported in this study prompt us to reconsider what “contextual” really means. The use of concrete materials and student discussion such as that recommended by Quinlan et al. (1987) and Lowe et al. (1993), and also reflected in algebra games, was sufficient to engage and help students make sense of algebra processes.

The results support the notion that the essence of learning algebra like that of arithmetic is to make connections between materials, patterns and symbolic meaning through the medium of language (e.g., Booker et al., 2001). In this instance, the use of materials was guided by resources that have been available to Australian teachers since the late 1980s (e.g., Quinlan et al., 1987) and early 1990s (e.g., Lowe et al., 1993). These resources place emphasis on students making meaning through the use of materials, discussion and students’ articulation of their mathematical thinking, through natural language initially, then subsequently through the specialised language of algebra conventions. The results support the explicit teaching of the nuances and processes of algebra in an algebraic and symbolic setting (e.g., Kirschner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1999). The findings should encourage teachers and researchers to look again at multiple representational techniques and the use of concrete material resources as an alternative to the way algebra is traditionally taught in middle school.

An examination of student needs in needing the links between representations to be made explicit throughout the trial and, to a less extent the error patterns exhibited in the final test, indicate that much of the “trouble” for students was not associated with algebra but rather had its roots in incomplete understanding of arithmetic structures. The error patterns associated with doing operations with integers (operating with negative integers), lack of understanding of the equal sign, over generalisation of cancelling procedures (fraction errors), and an incomplete understanding of the distributive law, have their roots in arithmetic misconceptions, and incomplete understandings and inability to transfer arithmetic understandings to algebraic contexts.

In this small and “streamed” class most of the misconceptions usually could be addressed through the intervention of the teacher and researcher. Subsequent to this analysis, the use of more explicit linking of arithmetic and algebraic structures will be investigated in future iterations of the research study (e.g., the application of the distributive law in two digit multiplications and expansion of algebra expressions). In a larger and heterogeneous class it is easy to envision that a limited understanding of the structures of arithmetic and inability to see their relevance to algebra could spell the end of algebra competency and confidence among students. We concur with the assertions of previous authors (e.g., Lins & Kaput, 2004; Warren & Cooper, 2006) that critical concepts underpinning algebra (e.g., equal concepts, integer study, fractions, the distributive law and general arithmetic computational competency) need to be emphasised in the primary years. For example, younger students can be taught with the aid of materials in order to help them solve simple equations (Warren & Cooper, 2006). This process helps students understand the structures of arithmetic in that the unknown is seen as a quasi variable to be solved by backtracking, or arithmetic operations based about the balance model, and reverse operations that emphasise the meaning of equals. With the careful use of materials the balance model thinking can be extended to understanding how to solve equations with variables on both sides.

With an understanding of arithmetic, upon the beginning of formal algebra study, when arithmetic processes including “do the same to both sides”, “use a graph”, “guess and check”, and “backtracking”, do not work (Stacey & MacGregor, 1999), students would be equipped with an operational and structural understanding of arithmetic such that they can transfer the understanding to the “operational” then “structural” phases of algebra, and to “value” the study of algebra. The importance of valuing algebra is that usually arithmetic means do not work efficiently with “real algebra” problems, whereas algebra enables an efficient solution to be found (Stacey & MacGregor, 1999).

References

- Barrington, F. (2006). *Participation in Year 12 Mathematics Across Australia 1995-2004*. Melbourne International Centre of Excellence in Mathematics and Australian Mathematical Science Institute. University of Melbourne. Available: www.boardofstudies.nsw.edu.au/manuals/pdf_doc/maths_st6_lit_curr_rev_pt1.pdf
- Bazzini, L., Boero, P., & Garuti, R. (2001). Moving symbols around or developing understanding: the case of algebraic expressions. In M. den Huevel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 121-128). Utrecht, The Netherlands, PME.
- Bereiter, C. (2002). Design Research for Sustained Innovation. *Cognitive Studies, Bulletin of the Japanese Cognitive Science Society*, 9(3), 321-327.
- Bernstein, B. (2000). *Pedagogy, Symbolic Control and Identity*. (Rev. ed.). New York and Oxford: Rowman & Littlefield Publishers, Inc.
- Booker, G. (1987). Conceptual obstacles to the development of algebraic thinking. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the eleventh international conference Psychology of Mathematics Education* (Vol. 1, pp. 275-281). Montreal: PME.

- Booker, G. (2000). *The Maths Game: Using Instructional Games to Teach Mathematics*. Wellington: New Zealand Council for Educational Research.
- Booker, G., Bond, D., Sparrow, L., & Swan, P. (2004). *Teaching Primary Mathematics*. Frenchs Forest: Pearson Education.
- French, D. (2002). *Teaching and Learning Algebra*. London: Continuum.
- Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 59-78.
- Kaput, J. (1987). PME XI Algebra papers: A representational Framework. In J. Bergeron, N., Herscovics, & C. Kieran (Eds.), *Proceedings of the eleventh international conference Psychology of Mathematics Education*. (Vol. 1, pp. 345-354). Montreal: PME.
- Kaput, J. (1995, October). *A research base supporting long term algebra reform?* Paper presented at the Seventeenth Annual Meeting for the Psychology of Mathematics Education (North American Chapter). Office of Educational Research and Improvement, Washington, D.C.
- Kemmis, S., & McTaggart, R. (2000). Participatory action research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 567-606). Thousand Oaks, CA: Sage.
- Kieran, C., & Yerushalmy, M. (2004). Research on the role of technology environments in algebra learning and teaching. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 99-152). Boston: Kluwer.
- Kissane, B. (1999). Graphics calculator and algebra. In E. B. Ogena & E. F. Golla (Eds.), *8th South East Asian Conference on Mathematics Education, Technical Papers: Mathematics for the 21st Century* (pp. 211-219). Manila: Ateneo de Manila University.
- Kirshner, D., & Awtry, T. (2004). Visual Salience of Algebraic Transformations. *Journal for Research in Mathematics Education*, 35(4), 224-257.
- Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra. The 12th ICMI Study*. (pp. 73-96). Boston: Kluwer.
- Lowe, I., Johnson, J., Kissane, B., & Willis, S. (1993). *Access to Algebra: Book 2*. Carlton: Curriculum Corporation.
- MacGregor, M. (2004). Goals and content of an algebra curriculum for the compulsory years of schooling. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 313-328). Boston: Kluwer.
- MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation. *Educational Studies in Mathematics*, 33, 1-19.
- National Council of Teachers of Mathematics, Algebra Working Group. (1998). A framework for constructing a vision of algebra: A discussion document. In *National Research Council, The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium*, May 27 and 28, 1997 (Appendix E, pp. 145-190). Washington, DC: National Academy Press. Available on the Web at <http://books.nap.edu/catalog/6286.html>.
- Quinlan, C., Low, B., Sawyer, T., White, P., & Llewellyn, B. (1987). *A concrete Approach to Algebra*. N.S.W. Darlinghurst: Algebra Research Group.
- Sfard, A. (1994). The gains and pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.
- Sleeman, D. (1986). Introductory algebra: A case study of student misconceptions. *Journal of Mathematical Behavior*, 5(1), 25-52.
- Stacey, K., & Chick, H. (2004). Solving the problem with algebra. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 1-20). Boston: Kluwer.
- Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal*, 1; 24-38.
- Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher*, 90(2), 110-113.
- Thompson, S., & Fleming, N. (2003). *Summing it up: Mathematics Achievement in Australian Schools, TIMSS 2002*. Australian Council for Educational Research. Camberwell. Available online: http://www.timss.acer.edu.au/documents/TIMSS_02_Maths_ES.pdf.
- Van de Walle, J. (2006). *Elementary and Middle School Mathematics: Teaching Developmentally*. Boston: Pearson International.
- Warren, E., & Cooper, T. J. (2006). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, 17(1), 58-72.