The Case of Disappearing and Reappearing Zeros: A Disconnection Between Procedural Knowledge and Conceptual Understanding

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We report on 25 Year 5-6 students' written responses to two items taken from an assessment of mental computation fluency with multiplication, alongside their reasoning of the strategy they had employed, which may or may not have made use of the associative property. Coding of this interview data revealed four distinct levels of conceptual understanding of the associative property, which teachers could use to inform their planning. The findings reveal the complexity associated with assessing multiplicative mental computational fluency and students' reliance on procedures often considered by them to be more magical than logical.

When assessing students' mental computational skills, fluency is often associated with accuracy and efficiency, rather than students' understanding of the properties associated with accurate and efficient strategy use, such as the distributive property and associative property. However, a more comprehensive construal of fluency for assessment purposes appears warranted, given contemporary definitions view fluency as:

> the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another (NCTM, 2014, p.1).

According to Carpenter, Levi, Franke, and Zeringue (2005), procedural fluency includes being flexible in choosing how and when to use a procedure, but also encompasses aspects of relational thinking, and what Skemp (1976) described as "knowing what to do and why" (p. 86). Such definitions emphasise the interplay between procedural and conceptual knowledge for developing fluency, and exist in juxtaposition to the notion that fluency is about applying rote-learnt procedures, that may in fact interfere with developing conceptual knowledge (Mack, 2001).

Several studies emphasise the importance of understanding how students think about mental computation, and the strategies they use (e.g., Clarke, Clarke, & Roche, 2011). An interview is a powerful tool to gain insights into student thinking and to challenge any entrenched procedures or misconceptions that may otherwise be overlooked; as Hurst (2018) found when exploring students' procedural knowledge and conceptual understanding of multiplicative thinking.

In this study, our aims were to explore how students explained a strategy that made use of the associative property in a mental computational task, and to investigate the extent to which they understood this principle.

Background Literature

To situate the study, we briefly review the research literature related to associativity in the context of multiplication, and the issue of zeros in multi-digit computation.

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Student understanding of the associative property

Hiebert and Grouws (2007) indicated procedural fluency and conceptual understanding be developed concurrently, with conceptual understanding informing the use of procedures. In other words, students who have strong conceptual understanding can apply these properties (e.g., associativity, distributivity and commutativity) to derive and predict unknown facts from known facts, and solve problems more quickly and with greater flexibility in their thinking, than those who are yet to develop conceptual understanding (Dowker, 2004).

Indeed, several studies indicate that when young children learn about the operations it is essential that they not only learn about number facts and algorithms but also develop a conceptual understanding of the relevant underlying mathematical properties, which in the context of multiplicative concepts include commutativity, distributivity, and associativity (e.g., Hurst, 2018; Larsson, Pettersson & Andrews, 2017). However, there is little research related to students' understanding of associativity for multiplication compared to commutativity and distributivity.

Having an understanding of the associative property provides students with a range of efficient mental and written strategies, flexibility in their thinking, as well as knowledge of the structure of arithmetic (Warren & English, 2000). Carpenter, Franke, and Levi (2003) described the associative property as follows, "When you multiply three numbers, it does not matter whether you start by multiplying the first pair of numbers or the last pair of numbers" (p. 108). Symbolically it is represented as (ab)c = a(bc), or as an arithmetic example such as (2×4)×25= 2×(4×25). This property underpins the doubling and halving strategy. For example, to solve 16×25 a student can use their knowledge of doubling and halving to change it to 8×50, and then to 4×100. Essentially they are transforming the original problem into 4×2×2×25 using factorisation, and then applying the associative property to change the order of calculation (4×2×2)×25 to 4×(2×2×25) (Larsson et al., 2017).

However, student knowledge of the associative property is often superficial or under-developed. Warren and English (2000) found that the majority of Year 6 students in their study had poor facility with the associative and commutiative properties including not being able to explain them. Similarly, Hurst (2017) indicated that most students in Years 5 and 6 had some knowledge of the properties and named them, but could not articulate an understanding, or make connections between them. Thompson (2008) suggested that teachers might themselves have limited understanding of the associative property. As a result, they may teach these procedures without drawing attention to the underlying properties, or assist students to reason about the strategies they use.

Issue of zeros in multi-digit mental computation

A recurring theme in the literature relating to students' computation of multi-digit multiplication involving zeros is that many students appear to ignore the zeros, or remove them and add them back at the end of the computation, without understanding how these steps relate to the associative property (e.g., Hurst, 2018; Keiser, 2010; Swan & Bana, 2000). For example, Swan and Bana found that students who used the strategy of crossing off and adding zeros did not appear to have any understanding as to why the strategy worked. When Year 5 and 6 students were asked how they would solve 400×23 some indicated they would do four times 23 which is 92 and then just added two zeros (Hurst, 2018). Similarly, when asked to solve 26×45, one Year 5 student recognised that 26×45 is the same as 13×90, and that 13×9 would be a quicker calculation, then add a zero on the end to get the answer to 13× 90 (Keiser, 2010).

In each of the above studies, students were using (what might be considered) an efficient strategy for estimating two-digit multiplication problems called *truncation*, which is a strategy commonly taught to students. Truncation involves ignoring the ones digit and multiplying the tens digits and then adding two zeros to the product (Star, Rittle-Johnson, Lynch, & Perova, 2009). Others referred to this as the 'add zero' rule when multiplying decade numbers such as 2×80, by first doing 2×8=16 and then adding a zero, rather than seeing it as 2×8×10 (Carpenter et al., 2003). However, applying this rule does not necessarily promote procedural fluency (Star et al., 2009).

One of the purposes of the current study is to add to this literature by examining whether the use of truncation as a strategy does in fact reflect a deficient understanding of the associative property, or whether students use truncation as a shortcut, however still possess the underlying conceptual understanding.

Methodology

This study is part of a larger project that investigated mental computational fluency with addition and multiplication in Years 3-6. The focus of this small study is on Year 5 and 6 students' responses in an interview about their understanding of another student's mental computation strategies.

Participants: There were 25 students (13 females; 12 males) in Years 5 (n = 12) and Years 6 (n = 13) from a single school in the Eastern suburbs of Melbourne, Victoria. In Victoria, students typically turn 11 years old in Year 5, and 12 years old in Year 6. The school was relatively socio-economically advantaged, with an ICSEA of 1141. Moreover, NAPLAN data indicates that this Year 6 cohort performed *above* similar schools and *substantially above* all schools on the Numeracy component in Year 5.

Measures: Items taken from the Mental Computational Fluency Measure – Multiplication (MCF-M) inform the current study. The MCF-M is a 30-item measure of mental computational fluency. The MCF-M is similar in structure to the Mental Computational Fluency Measure – Addition, which has been described previously, and demonstrated to have excellent internal consistency (α = 0.92; Russo & Hopkins, 2018).

The MCF-M requires students to think from the perspective of a fictional student named Emma. At the beginning of the assessment, students were given these instructions:

> *Emma is good at multiplying numbers. She uses clever strategies to make multiplication easier. Your job is to try and think like Emma did. Explain what Emma did to get the number in the box.*

The two items of relevance to the current study are presented below. The items chosen represent Emma's use of the associative property for multiplication.

Item no.	*Emma thought…*	*What did Emma do to get …*
1	*60 × 90* *is the same as* *54 × [100]*	*[100]?*
2	*800 × 70* *is the same as* *56 × [1000]*	*[1000]?*

Figure 1. Two associative property items from MCF-M.

Data collection: As part of a larger study, 100 Year 5 and Year 6 students from this school completed the MCF-M. Following completion of this instrument, 25 students were selected to participate in follow-up interviews in relation to five of the items (two of which are described in this paper). Student selection was based predominantly on convenience (that is, they finished the MCF-M earlier than the allocated 40 minutes), however an attempt was made to ensure that a variety of students were interviewed (e.g., a balance of Year 5 & Year 6 students; and of males & females). The purpose of the interviews was to gain insights into the students' interpretation of Emma's thinking. Doing so provided us with insights into the students' own understanding of the associative property. The student interviews were between 2.6 and 7.1 minutes in duration (mean = 5.1 minutes). Interview questions included:

- Explain how you figured out what Emma was thinking for this question?
- Did you have to change your thinking to think like Emma? In what way?
- How would you have done this one?

Data analysis: Grounded theory (Corbin & Strauss, 2015) was used to develop guidelines for classifying student interview responses. Initially, student written responses to the items were coded as correct or incorrect. Next, the first two authors independently read through the interview transcripts, using open coding to identify key themes related to each student's mathematical thinking. In collaboration, the authors undertook an additional cycle of coding, refining interpretations and categories. It was during this additional coding cycle that it became clear that these categories represented a continuum of conceptual understanding of the associative property (see Figure 2). Exemplary quotes illustrating each of the categories are included in the results and discussion section.

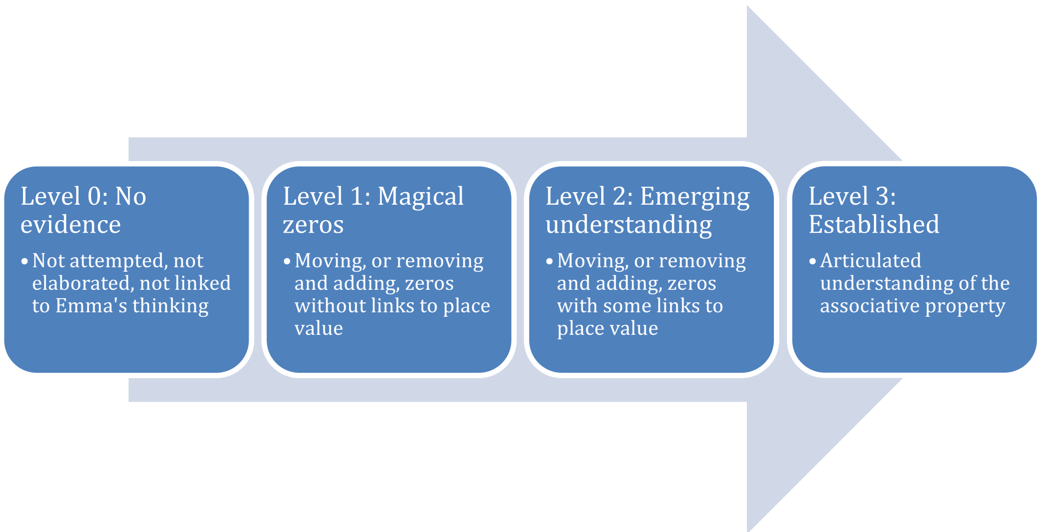

Figure 2. Continuum of understanding of the associative property.

Results and Discussion

In this section, we present the results of the initial analysis of the student responses to the two assessment items, followed by illustrative examples of their interview responses, with particular focus on the student responses that were coded as 'magical zeros'.

Table 1 shows the responses of the cohort [correct (C), incorrect (IC) not attempted (NA)] and the classification of their explanations as determined by the analysis. If students had an incorrect response for the first item, it was likely they would use similar thinking for the second item, or not attempt it.

Table 1.
Responses for each item (n=25) and classification.

Student	Year Level	Item 1 60 × 90 same as 54 × 100	Item 2 800 × 70 same as 56 × 1000	Classification
Amy	5	IC	NA	L0- no evidence
Linda	6	C	IC	L0- no evidence
Leo	6	IC	IC	L0- no evidence
Mindy	6	IC	NA	L0- no evidence
Artie	5	IC	IC	L1- magical zeros
Chad	5	IC	IC	L1- magical zeros
Emma	5	IC	NA	L1- magical zeros
Everly	6	IC	NA	L1- magical zeros
Ho	5	IC	IC	L1- magical zeros
Hugh	5	IC	IC	L1- magical zeros
Jim	5	IC	IC	L1- magical zeros
Kai	6	IC	IC	L1- magical zeros
Ken	5	IC	IC	L1- magical zeros
Lauren	6	IC	IC	L1- magical zeros
Mike	5	IC	IC	L1- magical zeros
Omar	6	IC	IC	L1- magical zeros
Richie	6	C	C	L1- magical zeros
Ted	6	C	C	L1- magical zeros
Annika	5	IC	IC	L2- emerging
Millie	6	IC	IC	L2- emerging
Sandra	6	C	NA	L2- emerging
Terry	5	C	IC	L2- emerging
Carrie	5	C	C	L3- Established
Jye	5	C	C	L3- Established
Paul	6	C	C	L3- Established

The results indicate that, of the 25 students who completed the assessment, only eight students correctly responded to Item 1 and five students correctly responded to Item 2. Five students who were successful on the first item were also successful on the second.

In relation to their interview responses, more than half (14 or 56%) were coded as Level 1: 'magical zeros', whereas only three students' responses (Carrie, Jye, Paul) were coded as Level 3: 'established' (having an understanding of the associative property). Interestingly, the interview responses of some students whose initial responses were coded as correct used 'magical zeros' thinking (e.g. Richie, Ted), or 'emerging' thinking (Sandra, Terry). Three students (Linda, Sandra, Terry) who correctly interpreted Emma's thinking for Item 1, were either incorrect or did not attempt Item 2. In the interview however, both Linda and Terry used the same thinking for Item 2 indicating their thinking was consistent.

The following are examples of the students' thinking, as classified by the authors.

Level 0: No evidence: Neither responses explained the 100 in the given context.

> I just knew that 10 times 10 is 100. (Linda)
>
> Did 6 x 9 is 54 then timesed 54 by 100. (Leo)

Level 1: Magical zeros: Both responses indicated the removal and return of the zeros.

I would have done 6 x 9 and then I would have put the two zeros on the end. (Richie)

Do 6 x 9 first and leaves 2 zeros, they make 100 because 100 has 2 zeros. (Anika)

Level 2: Emerging: Both responses indicate some understanding of place value.

So like, six times nine equals 54 here, then there's two zeros left, which there are two zeros in 100, so then she got 100 by doing that. They (the zeros) are from like the 10s I think. These are the 10s of the hundreds. So it's like the place value. (Millie)

I think it was like 6 times 9 equalled 54 and then you'll still have the two 10s, times them together would equal 100. (Terry)

Level 3: Established: Responses indicated an understanding of the associative property.

I know that if you do - basically 60 is 6 times 10 and then 90 is also 9 times 10 and that means if you do 6 times 9 that's 54. That leaves 10 here and 10 there, which is 100. (Jye)

For 800 x 70 she did 8 times 7 is 56 but then, instead of 10 times 10, because it's 800, there's 100 times 10. (Carrie).

The surprising and unexpected result was the proportion of students who used what we classified as 'magical zeros', and that for many students this type of thinking was the norm when engaging in mental computation, which was only revealed in an interview situation. The following discussion focuses on those students whose thinking was classified this way.

Students who adopted 'magical zeros' thinking used predominantly spatial language when discussing how zeros on the end of a multiplicand or multiplier could be manipulated, to aide with mental computation. For these students, zeros in this context took on a tangible quality, which seemed largely independent of the conceptual mathematical relationships in which they were embedded. For example:

There are two zeros right there for you. (Lauren).

You take out the two zeros… put them to the side… then you just add them both on. (Richie).

Several of these students employed this spatial language in the context of explaining their application of a procedure that they had been explicitly exposed to as being mathematically permissible. This procedure was often viewed as a fundamental rule that students did not challenge or question, even if they did not understand it:

Well in maths you can do that… I normally move the zeros, that's normally easier for me... I don't know how to explain it. (Ho).

So I just got told if there's zero left, you just add the one and then you put the zeros there. (Omar).

Because there's two zeros are already there, so you can't leave the zeros out of the sum, because then it wouldn't equal the right answer. (Mike).

The two zeros are still there, they can't just be chucked away. (Ted).

To exemplify how entrenched this 'magical zero' thinking is, we have included below an excerpt from Lauren's interview. The questioning technique employed was intended to challenge her thinking, in particular, about 'where the 100 came from':

L: She times '6 x 9' to get the result of 54, and then she put the two leftover zeros and then she put a 1 at the front so it was 100 and then you just times '54 x 100', and if you do that you can just add two more zeros, so it's 5400… there are two zeros right there for you.

I: So you said she took the zeros off the 60 and the 90, then what does she do with them?

L: She put them with another 1, which made 100.

I: I'll stop you there, where did the 1 come from?

L: Because 60 and 90 are two digit numbers and it would be more than a double-digit number and there are two zeros right there for you.

I: Okay, so she put a one in front of the zeros but I am not sure where the one came from?

L: The one came from – I am not quite sure but the only thing I can think of is they're two digit numbers and that might be why.

I: So it is something to do with two digit numbers, but you are not sure. So how would you have solved 60 x 90?

L: I would have just done that but with the 54, I would not times it by 100, I would just add the extra zeros to make it easier.

This excerpt highlights several aspects of Lauren's place value understanding in particular her lack of understanding of quantity value. It is evident that these students have been taught the 'add zero' rule when multiplying decade numbers, but do not have an understanding as to why the rule works (Carpenter et al., 2003; Hurst, 2018; Swan & Bana, 2000). So entrenched was this procedural knowledge, and possible belief that mathematics problems can be solved using facts and rules that even when challenged, the students were reluctant to question their thinking.

It should be acknowledged that applying a procedure that involves manipulating zeros is perfectly acceptable, if underpinned by a conceptual understanding of the associative property. For example, when Jye and Carrie, two students categorised as Level 3, were asked how they themselves would have calculated the problems presented, both described truncation, that is 'removing and adding back the zeros', because this approach was less laborious than what Emma had done. However, the important point is that these students understood Emma's more formal use of the associative property, as they could describe where the 100 (or 1000) had come from. The use of this heuristic for these students is not problematic, because it is not at the expense of conceptual understanding.

Concluding Comments and Implications

In this study we sought to investigate what sense students made of the associative property. We examined their use of certain procedures when reasoning about another student's mental computation strategies, and the extent to which conceptual understanding of the associative property underpins these procedures. As indicated in other studies (Carpenter et al., 2005; Larsson et al., 2017; Perek & Kirshner, 2000), an understanding of the associative property is important to students' understanding of multi-digit multiplication and students' ability to work flexibly with numbers, as well as their understanding of the structure of arithmetic. Findings from this study raise questions about current instructional practices, and suggest that the teaching of rules without reason (Skemp, 1976) is still a contemporary issue. Furthermore, interviews from this study provided deep insights into students' conceptual understanding of the associative property. The continuum that evolved from the analysis of student responses adds to the research literature and provides teachers with a guide when assessing students understanding, to inform their planning.

We will conclude on a cautionary note. Although in the majority of instances, truncation as a strategy is indicative of 'magical zero' thinking, this was not universally the case. Consequently, just as we cannot infer the presence of conceptual understanding from the use of a particular procedure, interviews with Jye and Carrie suggest that it is equally problematic to conclude without further evidence that the use of a particular procedure reflects a lack of conceptual understanding. This is even the case if the procedure is not soundly grounded in mathematical principles.

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