The Tattslotto question: Exploring PCK in the senior secondary mathematics classroom

Nicole Maher
University of Tasmania
<Nicole.Maher@utas.edu.au>

Helen Chick
University of Tasmania
<Helen.Chick@utas.edu.au>

Tracey Muir
University of Tasmania
<Tracey.Muir@utas.edu.au>

Pedagogical Content Knowledge (PCK) is a powerful construct for examining the complexity of teacher knowledge. Together with teachers' moment-by-moment choices of action, it provides insight into teachers' knowledge and its influence on student learning. This paper investigates the PCK experienced by a senior secondary mathematics class during a lesson on probability. Data were gathered through observation, and student and teacher interviews. Multiple aspects of PCK were evident and were used in complex and dynamic ways.

This study is from a wider investigation of *pedagogical content knowledge* (PCK) at the senior secondary mathematics level. PCK has become a powerful construct for examining the complex relationship between content and teaching (e.g., Ball et al., 2008). PCK is an intricate blend of content and pedagogy, described by Shulman (1986) as knowledge that embodies those qualities of the content "most germane to its teachability" (p. 9). There has been little research into PCK at the senior secondary mathematics level, with only a few small studies (e.g., Maher et al., 2015; Maher et al., 2016) exploring the complexity of teachers' PCK and its relationships with broader contextual factors. The present study builds on this research by examining the moment-by-moment enactment of a senior secondary mathematics teachers' PCK during part of a lesson, and how this is perceived by the students. Data were collected from a lesson observation, a post-lesson interview with the teacher, and from students' perspectives on their teacher's knowledge and actions. This paper will explore the following research questions: *What aspects of PCK does a teacher of senior secondary mathematics display while demonstrating a worked solution? What do multiple sources of evidence of PCK reveal about teaching and learning during a teacher's worked solution to an item?*

Review of Literature

In the past 35 years, PCK has received considerable attention in the mathematics education research community (e.g., Hill et al., 2008). The appeal of PCK may be attributed, in part, to its potential to more precisely describe teacher knowledge in action (Gess-Newsome, 2015). While "teacher knowledge in action" refers to important practices such as the preparation of meaningful explanations in predictably challenging content areas, it does not necessarily concentrate on what it means to "teach effectively moment by moment" (Mason & Davis, 2013, p.186). It is posited that teachers' moment-by-moment pedagogical choices of action are potentially the most influential source of insight into mathematics teacher knowledge (Mason & Davis, 2013; Mason & Spence, 1999). Mason and Davis (2013) pinpoint the vital role of the "connective tissue" between *mathematical awareness* (e.g., noticing an absence in understanding from a learner) and *in-the-moment pedagogy* (e.g., having an appropriate pedagogical action come to mind when needed).

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Several frameworks have been developed to identify aspects of teacher knowledge including PCK (e.g., Chick et al., 2006; Hill et al., 2008). The Chick et al. PCK framework (2006) used in this study provides a detailed inventory identifying key elements of PCK, designed for observing teacher knowledge in action in the classroom. The framework has been applied to the work of mathematics teacher educators (e.g., Chick & Beswick, 2017), and within the context of secondary mathematics teaching. Vale and her colleagues (e.g., Vale, 2010) have used it to examine the mathematical knowledge of out-of-area mathematics teachers.

The elements of the framework offer a set of filters through which to explore teaching in action. The framework reflects the complexity of PCK, by identifying its components under three broad categories: "clearly PCK", "content knowledge in a pedagogical context", and "pedagogical knowledge in a content context". These categories represent the varying degrees to which content and pedagogy are intertwined rather than specifying sharply defined boundaries. Space prevents the inclusion of the entire framework but brief descriptions of selected PCK elements specific to this study are given in Table 1.

Table 1
Excerpts from a Framework for Pedagogical Content Knowledge (from Chick et al., 2006)

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses typical/likely student thinking about mathematics concepts (either generally or with reference to specific students).
Cognitive Demands	Identifies aspects of the task that affects its complexity.
Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Structure and Connections	Makes connections between mathematical concepts and topics, including interdependence of concepts
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning (e.g., justifies an activity as developing understanding of long-term probability).

Methodology

This paper uses data from a wider investigation into PCK at the senior secondary mathematics level and explores aspects of PCK from the perspectives of a teacher, his students, and the researcher, by examining an episode from one lesson. A Year 11/12 Mathematics Methods class from a Tasmanian independent school participated. Mathematics Methods is the main pre-requisite mathematics course offered for students who intend to pursue tertiary studies in areas such as science and engineering.

The participants in the present study were the teacher, Mr McLaren, who had taught Mathematics Methods for 12 years, and the nine 16-18-year-old students in his class, five of whom provided data. Teacher and student names are pseudonyms. Data were generated during part of a lesson where Mr McLaren provided a worked solution to an item involving the practical application of a concept the students were studying. The episode was observed, video-recorded, and transcribed in full. After the lesson, the participating students completed a short questionnaire that asked them: (a) What did you find to be the most helpful explanation, example, or strategy that your teacher used in today's lesson? And (b) What did it help you learn? At the end of the lesson, the students participated in a 15-minute semi-structured focus group interview during which they commented on aspects of the lesson that they perceived were particularly useful. Mr McLaren also participated in a 20-minute interview after the lesson, discussing his actions during the lesson episode discussed in this paper. Both interviews were recorded and transcribed in full.

Teacher actions were identified by the authors and aligned to relevant PCK elements of the Chick et al. (2006) PCK framework. The teacher and student interview and questionnaire responses were also analysed for further insight into teacher PCK.

Results and Discussion

This section begins with a description of the lesson scenario involving a number of aspects of teacher knowledge. Insights into Mr McLaren's PCK—as illuminated by multiple sources of data—are then discussed. The scenario focuses on Mr McLaren's demonstration of the solution to the Tattslotto problem in Figure 1.

In Tattslotto, your chance of winning first division is $\frac{1}{8145060}$. Find the number of games you would need to play if you wanted to ensure a more than 50% chance of winning first division at least once.

Figure 1. The Tattslotto problem (condensed from Hodgson, 2013)

The Tattslotto problem scenario

During the lesson, Mr McLaren had introduced his students to applications of the binomial distribution and chose to demonstrate the solution to the Tattslotto problem as an example of a problem where the probability is known and the number of trials (n) is unknown. He guided the students through the process of setting up the inequality to model the problem. He identified that winning first division Tattslotto involved a binomially distributed random variable X with n trials and probability of success $\frac{1}{8145060}$, that is, $X \sim \text{Bi}(n, \frac{1}{8145060})$. With a mix of focused questions and explicit direct teaching, he helped the students to recognize that, given the item stipulates winning first division *at least* once, the situation can be expressed as "one minus the probability of *not* winning first division [in n trials]". He thus established the inequality $1 - \Pr(X = 0) > 0.5$ using the fact that $\Pr(X \geq 1) = 1 - \Pr(X = 0)$. Mr McLaren then guided the class through the development of the inequality shown in Figure 2, by using the formula for the probability distribution of a binomial random variable X, given by $\Pr(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$ where ${}^{n}C_{x}$ represents the number of ways that x different outcomes can be obtained from n trials, p is the probability of success (in this case winning first division), and q the probability of failure (not winning first division; equal to $1 - p$). The students had been introduced to this formula during the previous lesson.

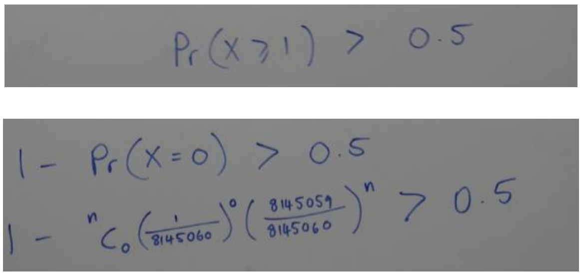

Figure 2. Initial stages in determining how many games are required in order to have a 50% chance of winning Tattslotto

After some procedural manipulation, which included dividing both sides by negative one and changing the sign of the inequality as a result, the inequality shown in Figure 2 was expressed as: $n \log_{e}\left(\frac{8145059}{8145060}\right) < \log_{e} 0.5$. Mr McLaren pointed to the inequality and asked the class, “What do we do now?” David suggested dividing both sides of the inequality by $\log_{e} 0.5$, so Mr McLaren reiterated that “we are trying to get n on its own, so we need to divide by the log of all that [points to $\log_{e}\left(\frac{8145059}{8145060}\right)$]. Now, is there anything else we need to know about?” There was a pause before Toby tentatively suggested that “The [inequality] sign changes”. Kale quickly retorted “No it doesn’t. I thought you said it only changes when you divide by a negative?” “That’s right, so why would the inequality change?” asked Mr McLaren. “It doesn’t” Kale persisted, looking puzzled. Mr McLaren assured them “It does change, but why?” Someone suggested, “because it’s a log” to which the teacher responded, “Yes, well, in a way because it is a log, but why?” David offered “Because there is a rule on our formula sheet?” Mr McLaren shook his head with a smile “No, there is no rule on your formula sheet”. He paused for a short while and then said, “OK, let’s have a look”. Mr McLaren began to write something on the white board but then quickly rubbed it off and changed tack. “OK, let’s think of any log. Now remember the log graph, this is the easiest way to look at it”. He sketched the graph of $y = \log_{a}x$ as shown in Figure 3.x

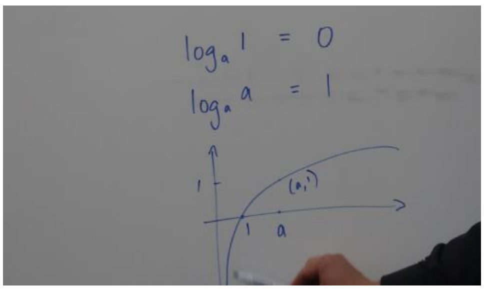

Figure 3. Mr McLaren’s sketch of the graph of $y = \log_{a}x$ used to show when $\log_{a}x$ is negative.

Mr McLaren highlighted the point at x=1 and Toby suddenly called out, “Oh, so that’s below one, so it’s a negative, so that’s why you change it around!” Mr McLaren nodded “Good, yes, any value of x less than one, or between zero and one, is negative”. He pointed to the region of the graph between *x*=0 and *x*=1 and reiterated that the logarithm to any base of any value for *x* between zero and one, in this case $\frac{8145059}{8145060}$, is negative. This was used to explain that when both sides of the inequality are divided by $\log_e\left(\frac{8145059}{8145060}\right)$, the inequality sign changes. "And it's a good thing too," Mr McLaren commented as he rubbed the board down, "otherwise we would find that we need to buy less tickets than we would actually need to buy. So, it's a good thing to look at what you're actually doing rather than just performing the calculations. OK, can someone evaluate that for me please?" [points to the right hand side of the inequality shown in Figure 4].

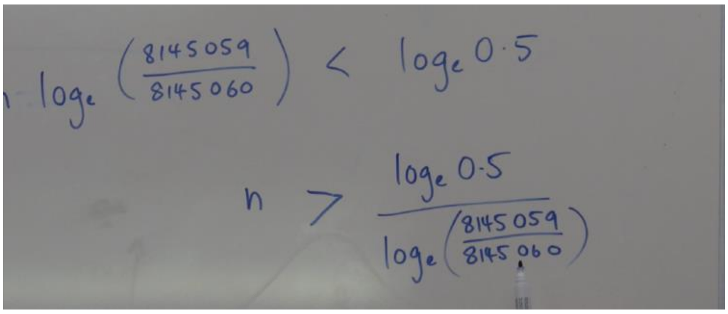

Figure 4. Final stages of the calculation of the inequality to determine the number of games.

Jonti performed the calculation, yielding 5645727.4. Mr McLaren asked "Can you buy 0.4 of a ticket? [The students shook their heads.] You would still write it to one decimal place, but for your final answer you would round up. You have to round up because if you go less than the 0.4 then you won't have greater than 50% chance of winning". Kale exclaimed, "So you'd need to buy that many tickets?!" Someone else added, "Just to have a 50% chance of winning once! What?" Mr McLaren smiled, "Yes, so you need to buy a lot."

Discussion of PCK

Multiple elements of PCK from the Chick et al. framework provided filters through which to examine the teacher's PCK in action in this episode. *Knowledge of explanations* was evident throughout Mr McLaren's worked solution. A combination of *knowledge of student thinking* and *knowledge of the cognitive demand of the task* were apparent, in that Mr McLaren was aware that students may not make the necessary connections with their previous work on logarithms to recognize that $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. These aspects of PCK were intertwined with *knowledge of teaching strategies,* evident when Mr McLaren posed strategic questions to encourage the students to make the connection between the value of the logarithm and the reversal of the inequality sign. As the students did not appear to make this connection by themselves, Mr McLaren sketched the graph of $y = \log_a x$, where "a" represents any base, to assist them to recognise that the value of $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. Mr McLaren's decision to sketch the graph appeared to be made in-the-moment, in that it seemed not to be something that was planned in advance, which highlights the complex and dynamic nature of teacher knowledge. During this in-the-moment event, Mr McLaren drew upon his own mathematical content knowledge and demonstrated several aspects of PCK including *representation of concepts, knowledge of mathematical structure and connections*, and *knowledge of the curriculum* (evident because the teacher drew upon his knowledge of where logarithmic graphs were placed within the course). Further evidence of Mr McLaren's PCK was provided in the post-lesson interview, as seen below:

Researcher: The reversal of the inequality sign generated a lot of interest. How did you come to decide on how to show them why the sign changes?

Mr McLaren: … It's a hard one to remember because it [$\log_e\left(\frac{8145059}{8145060}\right)$] doesn't look like a negative number but umm I suppose it strengthens their understanding of logarithms. They were not understanding; well, they hadn't made any connections at that point.

While Mr McLaren's comment provides further evidence of *knowledge of mathematical structure and connections* and *knowledge of student thinking* it does not offer specific insight into his in-the-moment decision to use the log graph to show why $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. On reflection, it may have been valuable if the researcher had phrased her question more carefully to probe for specific details about Mr McLaren's in-the-moment decision to draw the graph. Nevertheless, it is apparent that Mr McLaren's content and curriculum knowledge were sufficient for him to bring to mind (a) the reason for the change in sign and (b) a representation that would help students see why the value of the logarithm is negative.

Several students commented on the usefulness of the way Mr McLaren unpacked the solution to the Tattslotto problem, as indicated in their responses to the researcher's question about the teacher's useful explanations, examples, or demonstrations.

Jonti: The log one … [Toby concurs with "The Tattslotto one"]. It was good he kind of like decided on that Tattslotto question because it sort of recaps other things that we knew already so you go through it and refresh your mind on log laws and add the new layer of um technicality to it … Umm I don't know, it's just, well, it doesn't look that hard but then the way you've got to go around it with the logs and switching the inequality sign as you go through as well.

Researcher: Did you find anything in the explanation useful in helping you to piece it all together?

Jonti: Yeah umm I liked how he went through each step not like skipping over any one of them assuming you would know it.

Carl: Yeah umm just I kind of understood the thing except for getting tripped up when you've got to remember your log laws and like, and it was funny that you even had to draw a graph so go right back to the start to show us like if it's below like why you have to switch.

Jonti: The graph made it a lot clearer as to why you change the sign.

Kale also recorded that "the log explanation was the most useful because it explained and refreshed things for me like the log laws and changing < and >" (post-lesson questionnaire). These responses suggest that the students appreciated Mr McLaren's approach to solving the problem and that the graph had assisted them to realise why the inequality sign changed. Here the teacher's fluency across topics is a key part of his knowledge, and something that he wants to convey to students.

Mr McLaren also discussed the reversal of the inequality sign within the context of the problem, highlighting that it "makes sense because otherwise we would find that we would need to buy less tickets than we would actually need to buy". This aspect of Mr McLaren's PCK was identified as *purpose of content knowledge* because he alluded to the way the mathematics content may be used within the context of the Tattslotto scenario. This connection between the mathematics itself and the context of the problem resonated with Carl in the student focus-group interview:

Carl: Yeah, because I was sitting there and I was like why did you switch it because it wasn't dividing by a minus but then it's like no because if you think about it, it's common-sense you're not going to have to only buy a small number of tickets.

During his post-lesson interview, Mr McLaren identified his reasons for selecting the Tattslotto problem.

Mr McLaren: Umm probably more so from a non-maths kind of perspective to sort of demonstrate the futility of umm Tattslotto and the chances of winning umm that's probably the main reason why I chose that particular question. It wasn't so much a maths choice in that respect.

The teacher's response provides evidence of both *choice of examples* and *goals for learning* since he justified choosing the Tattslotto problem because it illuminates the very low probability associated with winning first division. One student, David, commented on this aspect of Mr McLaren's approach in the following written response:

> The Tattslotto question was the most useful. It helped me to find the number of games needed for a 50% chance of winning the game and how stupid gambling is. (David, post-lesson questionnaire)

It is noteworthy that teacher and student responses in relation to the Tattslotto problem focused on the involvement of logarithms during the solution process rather than on probability concepts per se. For example, although Mr McLaren solved the problem by setting up the inequality as 1 – Pr (not winning in *n* games), thus demonstrating that content knowledge, the significance of this probability technique was not evident in the other data sources in that neither teacher nor students mentioned this as a key learning outcome. This might mean that students were familiar with the technique, and that Mr McLaren had *knowledge of student thinking* to be confident that they could use it fluently, or, alternatively, this was not identified as a key learning point, which may reflect some shortcomings in PCK.

Conclusions

While this study is limited to one account of a lesson episode, it provides a detailed snapshot of the nature of a senior secondary mathematics teacher's PCK in action, and its influence on students from their perspectives. The level of detail and specificity afforded by the Chick et al. (2006) framework rendered it useful for examining the moment-by-moment teaching and learning interactions between Mr McLaren and his students. As such, the study contributes to the field of research into the complexity of PCK at the senior secondary level from multiple perspectives, including the researcher, the teacher, and his students.

Multiple elements of PCK were evident in the scenario, particularly those from the "clearly PCK" section of the framework, including *knowledge of student thinking, knowledge of the cognitive demand of the task, knowledge of teaching strategies,* and *representations*. *Mathematical structure and connections* from the "content knowledge in a pedagogical context" section of the framework, and *Goals for learning* from "pedagogical knowledge in a content context" section were also evident in the data.

The combination of PCK elements from across the framework provided insight into the nature of the interactions between Mr McLaren, his students, and the mathematics itself, highlighting the complex and dynamic nature of PCK. For example, Mr McLaren called upon his own content knowledge to decide on which action to take in order to make visible for the students why the value of the logarithm was negative. There were also interactions with the broader teaching and learning context, with Mr McLaren's expressed reason for choosing the Tattslotto problem being to illuminate the "futility of gambling" rather than the mathematics per se. On the other hand, some key probability ideas, such as the role of the complement, may have been underemphasised because of this focus.

Other post-lesson teacher interview data supported the researcher's observations of *knowledge of student thinking* and *mathematical structure and connections* but was limited in terms of providing insight into Mr McLaren's specific choices of action. The Chick et al. (2006) framework for analysing PCK, however, offers a level of detail and specificity that is potentially useful for examining what comes to a teacher's mind in moment-by-moment teaching and learning interactions.

The students' perceptions of Mr McLaren's actions gave useful insights into PCK and supported evidence from the other data sources. For example, Mr McLaren's representation of the log graph representing the relationship between the value of x and its logarithm was particularly noticed and appreciated by the students. Similarly, the connection Mr McLaren made between the mathematics involved in the reversal of the inequality sign and the reality of the number of games that would need to be played, was valued by some students.

. Further studies that investigate PCK in different senior secondary mathematics contexts with a particular focus on the moment-by-moment pedagogical choices of action would also add to the limited research in this area.

References

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Chick, H., Baker, M., Pham, T., & Cheng, H. (2006). *Aspects of teachers' pedagogical content knowledge for decimals*. Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Chick, H., & Beswick, K. (2017). Teaching teachers to teach Boris: A framework for mathematics teacher educator pedagogical content knowledge. *Journal of Mathematics Teacher Education, 21*(5), 475-499. https://doi.org/10.1007/s10857-016-9362-y

Gess-Newsome, J. (2015). A model of teacher professional knowledge and skill including PCK. In A. Berry, P. Friedrichsen, & J. Loughran (Eds.), *Re-examining pedagogical content knowledge in science education*. New York: Taylor and Francis (pp 28-42). https://doi.org/10.4324/9781315735665

Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372-400. https://doi.org/10.2307/40539304

Hodgson, B. (2013). *Maths quest 12 mathematical methods CAS* (2nd ed.). Milton, QLD: John Wiley & Sons Australia.

Maher, N., Muir, T., & Chick, H. (2015). Examining PCK in a senior secondary mathematics lesson. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins: Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 389-396). Sunshine Coast: MERGA.

Maher, N., Chick, H., & Muir, T. (2016). "I believe the most helpful thing was him skipping over the proof": Examining PCK in a senior secondary mathematics lesson. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research: Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australiasia* (pp. 421-428). Adelaide: MERGA.

Mason, J., & Davis, B. (2013). The importance of teachers' mathematical awareness for in-the-moment pedagogy. *Canadian Journal of Science, Mathematics and Technology Education*, *13*, 182-197. http://dx.doi.org/10.1080/14926156.2013.784830

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135-161.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Vale, C. (2010). Supporting "out-of-field" teachers of secondary mathematics. *Australian Mathematics Teacher*, *66*(1), 17–24.