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The Widening Gap—A Swedish Perspective

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Transition problems from secondary to tertiary level in mathematics have been a recurrent issue in Sweden. This paper summarises the development during the last decades. Results from two recent research studies that illuminate the transition problem are presented. The first one, based on empirical data from a major Swedish technical university, characterises the widening gap, in content and in approach, between secondary school and first year university courses. The second study deals with students' encounters with mathematical proof and is based on a large investigation at another main Swedish university. We discuss the influence on the current transition problems of school reforms and of the great expansion of higher education in Sweden during the last 10 – 15 years in view of the results from the research studies.

Introduction

The transition from secondary to tertiary level has been identified as a crucial period and accordingly addressed in research during the last decades (e.g. Hillel, 2001; Hoyles, Newman & Noss, 2001; Kahn & Hoyles, 1997; Kajander & Lovric, 2005; Wood, 2001). Many students encounter serious problems in their mathematics studies at tertiary level, and initial problems may lead to prolonged or unsuccessful studies or even cause students to quit their mathematics studies. In addition, weak recruitment to science and engineering and high drop-out-rates are documented or are assumed consequences.

Transition problems between secondary and tertiary level have been a recurrent issue also for Swedish mathematicians, mathematics teachers and policy makers. Several national commissions have reported on the situation during the last decade and they point to increasing problems and a widening gap during the last 10 to 15 years. Lately the transition problem has gained full attention from Swedish researchers. Results from systematic investigations into the situation have been presented (e.g. Juter, 2005; Thunberg, Filipsson & Cronhjort, 2006), among them a couple presented in doctoral theses (e.g. Hemmi, 2006). The studies involve new entrant students, as well as teachers at universities and upper secondary schools, and add considerable insight into the character of the transition problems.

The aim of this article is to clarify the nature of the widening gap in Sweden and to investigate possible reasons within the educational system for the increased problems. Our research questions are the following:

1. What are the main traits of the widening gap between upper secondary and university level concerning the mathematical content and the perception of mathematics?
   Specifically we investigate the following aspects:
   - The curriculum gap.
   - Perceptions of knowledge and learning of mathematics.
   - Students' encounters with proofs and proving.

2. What recent and historical development of the educational system may partly account for the aspects of the widening gap described in the first part of the article?

The article follows the structure above. We start by presenting two Swedish research projects illuminating some main traits of the transition problem. One is a study of the gap between upper secondary school mathematics curriculum and expectations of students' knowledge at a Swedish university of technology. The other deals with the culture of proofs and proving at upper secondary level and at university level, and it uses empirical data from the science faculty of another university. Both studies are based on data from main Swedish universities.

In the second part of the article, we provide a brief overview of the development of the Swedish educational system, with an emphasis on the last decade. Further, we describe the latest Swedish school reform and its possible impact on the transition problem. We also give examples of projects aiming to overcome the gap between upper secondary and university mathematics.

Finally, we discuss the results of the research studies and present possible connections between the general development of the educational system, the implementation of school reforms and the transition problems in the light of the results of the two research studies presented in the article.

Background

The most recent curriculum reform of compulsory and upper secondary school in Sweden came into effect in 1994, (National Agency for Education, 2006a, 2006b). All schools—whether independent or run by the local municipality—conform to the general curriculum and to the subject specific curriculum decided upon at national level. National evaluations and quality insurance of the school system are carried out through regular inspections of schools and through national written course-specific tests for selected courses. All students taking such a course participate in the national test.

Since 1994 the mathematics curriculum for all study programmes—theoretical as well as vocational—consists of one, two, three or more out of a series of common courses. There are in all five courses in mathematics, *A – E,* that build upon each other and national tests are given for all these courses.

Formal prerequisites for entering certain university programmes (like engineering programmes and teacher training programmes) are decided by the National Agency for Higher Education. Universities may lower (but not raise) these requirements for their own particular programmes. In an effort to attract more students some universities have lowered entering requirements from upper secondary course *Mathematics E* to the less advanced course *Mathematics D*.

The Curriculum Gap and Perceptions of Mathematics—Study 1

During 2004 and 2005 a study, headed by one of the authors of this paper, was carried out at the Royal Institute of Technology (KTH) in Stockholm, in which the transition problem was considered as a matching problem between two sequential educational systems (Thunberg & Filipsson, 2005a; Thunberg, Filipsson & Cronhjort, 2006). The aim was to compare the *goals and ambitions* of mathematics education in Swedish upper secondary school with the *expectations* of the new students held by the tertiary level. The same approach was recently taken in a British study, and structural problems similar to the Swedish ones were identified

(Hoyles, Newman & Noss, 2001).

A gap between secondary and tertiary level has been identified in several other studies from different countries. Results similar to the Swedish situation even on a detailed level can be found in recent studies from Canada (Kajander & Lovric, 2005) and the Netherlands (Heck & van Gastel, 2006). Similarly, transition problems in Hong Kong are described in (Hing, 2005). The situation in France and in Quebec some ten years ago is discussed by Guzman, Hodgson, Robert and Villani (1998). An international overview is given by Selden (2005).

Methodology

A series of investigations was conducted during the academic year 2004-2005. In this section we briefly describe the methodology. In order to facilitate cross-referencing, we will label the different investigations with capital letters A–F.

A. University expectations were identified from the literature and the exercise sets that students were encouraged to use for preparation during the summer and during a two weeks preparatory course right at the beginning of the first semester. By definition, this material was regarded as most important for new entrants from the university's point of view. It was intended as a revision of mathematics already studied in secondary school and presented in a condensed form. The material thus offered a description of tertiary level expectations, in terms of the central concepts, techniques and typical problems students were supposed to be familiar with (Thunberg & Filipsson, 2005b).

The new entrants' performance and pre-knowledge was compared to university expectations through various methods.

B. A group of approximately 100 new engineering students at KTH were asked to record and comment on their experiences of the preparatory course, especially noting when they encountered concepts, techniques or problem types that they found difficult or unfamiliar. Answers were received from 36 students representing different backgrounds from upper secondary school. Those who did answer had slightly higher grades in mathematics from upper secondary school than the whole group, thus students' difficulties might in fact have been underestimated by this part of the project (Thunberg & Filipsson, 2005c).
C. University teachers teaching the preparatory course were also asked to note their experiences of the new students' work with the course. Answers were obtained from 29 teachers out of group of 46 (Thunberg & Filipsson, 2005c).
D. In a questionnaire, secondary school mathematics teachers were asked to grade how well prepared typical students would be after graduation from upper secondary school to handle various typical problems from KTH's preparatory course. The questionnaire was sent to 90 teachers in the Stockholm area, answers were obtained from 19 of them, representing 10 different schools (Thunberg & Filipsson, 2005d).
E. Students' solutions to written exams at the university were examined closely, looking for typical mistakes and misunderstandings. Two sets of student solutions were examined. The first one consisted of about 50 corrected exams from the preparatory course (Cronhjort 2005). The

second set was taken from a first semester exam in one-variable calculus, where about 150 students' solutions to three selected problems were examined (Enström & Isaksson 2005).

The national tests in mathematics for upper secondary school were taken as an expression of the goals of the upper secondary level. The tests consist of two parts, a first part aiming at testing basic skills where calculators are not allowed and a second problem-solving part where calculators may be used.

F. Two complete sets of national tests for *Mathematics A–D*, those given during 2002 and 2005[1], were compared with university expectations as they appear in the summer revision material and the preparatory course at KTH (Thunberg, 2005).

The Curriculum Gap

The students' own comments, as well as their teachers' reflections on their difficulties with the preparatory course at KTH, clearly indicated the existence of a curriculum gap, consisting of pieces of mathematics content that the students are expected to be well familiar with when they start their studies at the university but which are not part of the upper secondary school curriculum. The picture was confirmed by the secondary school teachers' answers to the questionnaire. We now describe the main constituents of this gap.

*General knowledge about functions***.** From (A) one can see that *piecewise defined functions* and *composition of functions* are expected to be concepts familiar to the new entrants to the university, but from the answers in the questionnaire (D), we know that neither of them can be considered as well known to the typical student leaving upper secondary school. One university teacher in (C) also says that students' abilities to draw accurate sketches of elementary functions are below what is expected.

Analytic geometry. Although the Theorem of Pythagoras is, of course, well known to students in upper secondary school, few of them are familiar with the distance formula or the equation of a circle in its general form, since this is not part of the standard curriculum. When entering the university however the students are expected to be well familiar with the equation of a circle, and also to be prepared to generalise to general conics in the plane. Here is a typical exercise from the revision material given to the new students at KTH.

Determine the geometric meaning of the equation

$$x^2 + 2x + y^2 - 4y + 4 = 0$$

and sketch the corresponding curve. (From preparatory material used at KTH, 2004)

From the responses to the questionnaire to upper secondary school teachers (D), it is clear that not even students with good grades have the concepts and skills necessary for making sense of this exercise. In fact 14 teachers of 16 answering this particular question, classified this as an exercise that all students are probably completely unfamiliar with, while 2 teachers claimed that students with high grades might be able to follow a given solution, although they would lack the

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[1] National tests for *Mathematics A* are available at http://www1.lhs.se/prim/matematik/tidigare_kurs_a.html, and tests for *Mathematics B – D* can be found at http://www.umu.se/edmeas/np/information/np-tidigare-prov.html.

necessary tools to handle the problem themselves.

The absolute value function is, according to (D), not part of the upper secondary curriculum. In (B), many of the students mention equations and inequalities involving absolute values as a major difficulty; this is also noted by the university teachers in (C).

Inequalities. In upper secondary school students solve analytically linear inequalities in two variables, whereas nonlinear inequalities are solved graphically using the calculator. Still, the university expects students to be familiar with analytic techniques needed for problems like the following.

> Which values of x satisfy the inequality $x^3 - 3x^2 + 2x > 0$? (From preparatory material used at KTH, 2004)

In their answers to the questionnaire, the upper secondary school teachers state that even high performing students would typically not have the skills necessary to solve this exercise (D).

Our investigations also indicate misconceptions at the university level about how, and with what learning outcomes, certain topics are treated in upper secondary school. This includes algebraic techniques like "completing the square", techniques for solving special types of equations, and logarithms. We will discuss this further in the next section.

Differences in the Perception of Knowledge and Learning of Mathematics

There are different views on what it means to learn mathematics at secondary and tertiary levels. What aspects should be the focus? What are the desired outcomes of a learning process? These differences are most clearly visible in the analysis of mistakes and misunderstandings in students' solutions to examination tasks (E) and in the comparison of the requirements in national tests for upper secondary school, and the expectations from tertiary level (F).

Let us just say a word of caution before we go on: It is really not possible to identify a single common upper secondary attitude concerning these matters. There seem to be different subcultures with different views living in parallel in the upper secondary world. Below, we will compare university level expectations with requirements on national tests for upper secondary level. These national tests put great emphasis on testing students' understanding of concepts and of their problem solving skills, while computational skills and knowledge of formulas and mathematical theory is hardly required at all. On the other hand, recent research indicates that teacher-made tests typically have much more emphasis on computational skills (Boesen, 2006), and that most exercises in the more common textbooks are tasks that may be solved by a standard method presented on the page before (Lithner, 2000). Still, we find that typical weaknesses in university students' pre-knowledge do agree with areas not emphasised in the national tests.

From investigation A it is clear that at the university level, *routine skills in arithmetic and algebraic computations* are considered as an absolutely necessary ingredient when learning mathematics, and new entrant students are suddenly expected to handle much more complicated expressions and computations than they have met before. According to teachers at the university, students get confused when confronted with tasks where all necessary steps are not clear from the beginning (C). Computational errors of various kinds, often indicating a lack of routine, were also frequently observed in the student solutions to university examinations (E). We present below two examples of arithmetic skills required of new entrants at tertiary level, tasks that according to the upper secondary teachers (D) in fact are far from routine for many students.

Simplify the expression $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{5}}$. (From preparatory material used at KTH, 2004)

Compute $\dfrac{\left(3^{-4}\right)^{(-5)}}{243^{3}}$. (From preparatory material used at KTH, 2004)

In the upper secondary national tests in mathematics there are no tasks that require such computational skills. Instead, the focus is on problems that test students' understanding of numbers and arithmetic operations, like the following ones:

Place the numbers 25, 102 and 0.1 in the squares in a way that makes the following expression as large as possible $\dfrac{[\ ]-[\ ]}{[\ ]}$. (From national test for *Mathematics A*, 2005)

The most complicated numeric calculation required in any problem in any of eight given national tests during 2002 and 2005 appears in a problem where the last step in the computation of a definite integral leads to evaluation of the expression:

$\dfrac{3^3}{3}-3-\dfrac{1}{3}+1$ (From national test for *Mathematics D,* 2005).

When studying student solutions, we see that a lack of algebraic skills is a common problem for many newcomers at the university (E). Here is a typical exercise from the introductory material at KTH that many students would perceive as difficult (B).

Simplify $\dfrac{a^3+ab^2}{a^3-ab^2}\cdot\dfrac{a^2-ab}{a^2+ab}$. (From preparatory material used at KTH, 2004)

No problems on the national tests require algebraic skills at this level. Instead one finds:

Simplify $x(2x+5)-2(x+3)$. (From national test *Mathematics B,* 2005)

Use the conjugate rule to simplify $\dfrac{a+3}{a^2-9}$. (From national test *Mathematics C* 2005)

(In national tests, students have a table of formulas, including the conjugate rule, at hand).

Knowledge of elementary functions and standard identities. University expectations, according to (A), include a working knowledge of the logarithmic function and the laws of the logarithm, as is illustrated in the following exercise.

Solve the equation $\ln x+\ln(x+4)=\ln(2x+3)$ (From preparatory material used at KTH, 2004)

In the national tests, from 2002 and 2005, we find no tasks requiring this type of computation with logarithms. The only equation involving logarithms encountered appears in the last stage of solving an integral equation, where one has to determine a such that $\ln a - \ln 1 = \ln 2$ (from national test *Mathematics D,* 2002). This requires no computations; one only needs to know that $\ln 1 = 0$. In fact, logarithms in upper secondary school are mainly used in connection with growth problems as a tool to solve for the exponent using the calculator, and the algebraic properties of the logarithmic function are not stressed.

Tables and calculators. In first year mathematics at university, students are expected to work without calculators and tables. Routine skills and knowledge of formulas and theorems (procedural knowledge) are considered necessary for the understanding of concepts and theory, as well as important tools in problem solving. This seems to be in sharp contrast with the upper secondary level goals, at least as they manifest themselves in national tests, where calculations and formulas seem to be regarded as difficulties that hinder students from a deeper understanding, and thus should be avoided. This is clearly illustrated in the examples of this section. When analysing common errors and mistakes among the new entrant university students (E), one finds that a common mistake is the use of incorrect formulas. We propose the following explanation to this fact: in upper secondary school, students could always rely on tables with all useful formulas listed. There was never any need to internalise formulas as part of a larger whole or to develop strategies to conjecture, test and falsify or prove formulas. Perhaps formulas rather appeared to be rules codified in the tables and decided upon by some authority (the teacher, the book or some famous mathematician) for reasons not to be questioned or comprehended. A particular example of this would be *trigonometric functions,* which even though extensively studied in upper secondary school, still are considered as a major difficulty when entering university.

The findings in Study 1, presented above, can be summarized as follows. There is a definite curriculum gap between upper secondary school and tertiary level, consisting of concepts and techniques that are not a part of the upper secondary curriculum (neither the intended nor the attained) but still are expected as pre-knowledge by the university. This gap includes knowledge about, and skills with, elementary functions, some analytic geometry and techniques for handling polynomial inequalities, the absolute value function and certain algebraic equations.

There is also a cultural difference between the two levels in the perception of mathematical knowledge and mathematical learning. Compared to the upper secondary curriculum, university teachers expect students to have a higher degree of algebraic and numerical skills, more knowledge of and standard identities for elementary functions, and more ability to reason with and about formulas. We remark that the interplay between procedural knowledge and other competencies, like conceptual understanding or problem solving skills, is a subtle and difficult question. Interesting discussions relevant to the secondary-tertiary transition can be found in (Martin, 2000) and (Tall, 2000).

Although Study 1 focused on upper secondary schools in the Stockholm area and new entrants to KTH, the problems described above are quite general—this has been confirmed during a number of presentations of this investigation at Swedish authorities, universities and teachers' meetings. Indeed, as mentioned above the situation is also in many ways similar in other countries like Canada (Kajander & Lovric, 2005) and the Netherlands (Heck & van Gastel, 2006).

How Students Encounter Proof—Study 2

Proof has had a diminished place in the school curriculum of many countries during the last decades (Hanna, 1995; Niss, 1999). At the same time many researchers have focused on students' problems with proof and their encounter with more formal mathematics at university level (e.g. Chin & Tall, 2000; Juter, 2006; Moore, 1994; Nardi, 1996; Weber, 2001).

This section is mainly based on a doctoral thesis by one of the authors of this article (Hemmi, 2006). The aim of the thesis was to describe students' encounters with proof during their university-level mathematics study in Sweden. The students were aiming at further studies in natural science, economy, applied or pure mathematics or a teacher education. Here, we focus on the results about students' school background and its relation to their experiences about proof and students' difficulties with proof at university level.

Methodology

The data consists of survey responses of about 170 university new entrant students, transcripts of tape-recorded focus group interviews with students in various semesters, and of semi-structured interviews with 13 mathematicians (out of about 40 senior academic staff in all) in the mathematics department at Stockholm University. Protocols from the observations of lectures, interviews with experts as well as textbooks, syllabuses and other material are used as complementary data. The aim of the surveys was to obtain some information about students' upper secondary school background and about how they related to proof when they started to study mathematics at the university. The questionnaire consisted of multiple-choice questions, some open questions and statements with a five-point Likert scale, from fully disagree to fully agree.

Seven focus group interviews were organised during 2004-2005 among mathematics students at different levels of study: basic, intermediate or advanced[2]. The interviews were semi-structured and based on items that had been piloted with one student. The interviews lasted from one to two hours and were tape-recorded. The items focused on in the interviews were *students' upper secondary school experiences concerning proof*, *students' responses to the questionnaire* (if they had responded to it), *students' university experiences concerning proof* and *items from observed lectures*. The individual interviews with mathematicians took place in 2003-2004 and lasted from 30 minutes to two hours. The mathematicians were invited to reflect on items like *changes in the contents of undergraduate courses concerning proof, changes in students' prior knowledge concerning proof, how students meet proof in their lectures and lessons* and *why students should learn proof*. For details about the data analysis see Hemmi (2006).

Students' Background

According to the survey responses, students who start to study university level mathematics in Sweden have varied backgrounds with regard to their experiences with, and knowledge about, proof.

About one half of the students who responded to the surveys stated that their upper secondary school teachers proved statements once a week or every lesson. Even though many of these students stated that they had seen teachers' proofs, including derivations of formulas, very few of them had participated in the

[2] The *basic course* is taught during the first semester, the *intermediate course* during the second semester and the *advanced courses* during the third and fourth semesters.

practice of constructing proofs, according to the questionnaire responses and the focus group discussions. Responses to the question *"How often did you practice proving statements by yourself in upper secondary school?"* showed that over half of the Swedish students had very little such practice: once or twice a semester (19%); or even more seldom (40%). Yet, there was a small minority (7%) who stated they had practiced proving every lesson and also strongly agreed with the statement *I have had the possibility to practice proving by writing in school*.

Students' own investigations (alone or in groups) that can lead to hypotheses, or sometimes to proofs, seem to be unusual in the Swedish upper secondary school according to the students. Over 80% stated that they had had such activities only once or twice a semester, or more seldom (70%). However, there was a small minority (3%) who stated that they had worked in an investigative manner every mathematics lesson.

These findings were also supported by a textbook analysis that showed that tasks encouraging students to engage with investigations and conjectures were largely lacking in the upper secondary school textbooks (Nordström & Löfwall, 2005). Further, the space given to proving tasks in general in the textbooks is minimal (about 2%), compared to practical applications and routine exercises, with no rules for proving, no discussion of how proofs are created, and very few examples. However, there are some special mathematical domains where proving tasks are more common: in geometry, in the context of verifications of solutions of differential equations and verification of formulas of trigonometric functions.

> Example: Show that $2\sqrt{x}$ is a solution for the differential equation $2xy' - y = 0$.

According to the survey responses, students consider proof an essential part of mathematics, but see the act of proving theorems as more cumbersome than the act of computation. However, most of the students who responded related positively to proof; they wanted to learn more about it, and would have liked to have learned more about it in secondary school.

Changes in First Year University Courses Regarding the Treatment of Proof

All the mathematicians who were interviewed agreed that some changes had been made in course content with regard to the status of proof during the time they had been working at the department. To the question about possible changes in the treatment of proof one of them answered in the following way:

> [It is] devalued constantly, even if we still try to maintain a certain level, but it has been postponed. Earlier we had a theory part in the examinations of Analysis 1 and 2 [basic and intermediate courses] that were the same course then… We try to motivate some simpler theoretical things such as theorems about continuous functions and some other theorems where we only tell the students that a proof exists but we do not go through them. (Mathematician, 2004)

However, some of the mathematicians pointed out that the changes regarding the treatment of proof concerned only the lower level courses. The mathematicians had different views as to why the lower level changes, especially avoidance of proof, to the treatment of proof had taken place. They mentioned lack of time, students' lack of prior knowledge regarding proof, students' bad calculation skills, and students' lack of interest or fear of proof, new course literature, and economic aspects among the reasons for the changes in the status of proof in teaching undergraduate mathematics. The most common explanation was that the new entrants had little experience of proof from upper secondary school and thus, it was impossible to work with proofs in the basic course.

> Elements of proof in upper secondary school and in basic courses at the university have diminished, it's perfectly obvious. We have to adjust to the fact that the students usually have almost no experience when they come here. (Mathematician, 2004)

Contrary to this, students in the focus groups talked about experiences from the beginning of their university studies with mathematicians proving statements during the lectures, although they also mentioned that some mathematicians omitted long and technical proofs. Thus it is possible that many mathematicians do not consider a deductive presentation of mathematics as proving, while students do.

Students' Difficulties

Both the students and the mathematicians in the study talked about the difficulties students encountered when following proofs and when constructing their own proofs. Several mathematicians stated that students had difficulties with exact mathematical language. According to students, the language was different from the language that they were used to in their upper secondary school mathematics classrooms. Also the lack of experience with mathematical symbols was mentioned as a hindrance to students' capability to follow the presentation of mathematics.

> Now it's much more mathematical language, in upper secondary school it was possible to say everything with a little easier language and when the language becomes more difficult it's difficult to fully follow. You have to know these different symbols they write on the board, what it means. (Student – Intermediate course, 2004)

Another hindrance for students' engagement in the lectures seems to be the high tempo.

> They go through the things very fast in the lectures and if they are to prove something, for example logarithms, you have to be absolutely clear what a logarithm is and how it can be rewritten; all these rules, and if you don't do that you cannot follow […] I noticed that the tempo is much faster also with proofs, not so many comments on what the teacher does as in upper secondary school. (Student – basic course, 2004)

Sometimes gaps in the proofs and small, careless mistakes in the lecture notes can cause a lot of problems for students' understanding when they study the lecture notes at home. Such gaps were also observed during the lectures. Mathematicians usually seem to be in a hurry when they write proofs on the board. It is often difficult to follow and control that all the steps in proofs are correct. Also in the textbooks for the basic course, often some steps are left for the reader to justify. This can be hard for students who have no experience of proof.

Constructing One's Own Proofs

The students in the focus groups talked a lot about the difficulties they experienced when constructing their own proofs. They did not know how to start and they did not know what they could take for granted or when they had proved the statement. These difficulties were also identified and reflected by the mathematicians and they coincide with those Moore (1994) describes in his study of undergraduate students participating in a transition course.

According to the mathematicians, the proving tasks in the university examinations were always the most difficult ones for students. There were not many proving tasks in the examinations for the basic course, and between 2002 and 2006, only the following three tasks, from about 180, began with "Show that…".

1. Show that ${}^{a^c}\log(b^c) = {}^{a}\log b$ for all $a,b,c \in \{x \in R : x > 1\}$. (Basic course, 050531)
2. Show (preferably by using a Venn diagram) that if *A*, *B* and *C* are subsets of the complex numbers, then $(A \cap B) \cup (C \cap (A \cup B)) = (A \cup B) \cap (C \cup (A \cap B))$. (Basic course, 040312)
3. Suppose that $0 < a < b < c < d$. Show that $\frac{a}{d} < \frac{a+b}{c+d} < \frac{b}{c}$. (Basic course, 040108)

These few proving tasks represent different kinds of problems, and though mathematicians may consider them to be simple, they are not trivial for students. Students are often acquainted only with proofs of a certain type, where they have to show that the left hand side of an equality equals the right hand side by applying some known formulas. The tasks above are not of this type. They demand, for example, understanding of the role of definitions when proving statements. Such difficulties were mentioned both by some mathematicians and students.

> We never met theorems or definitions in upper secondary school. Sometimes I still have difficulties understanding the difference. I think that a theorem can look like a definition. (Student – intermediate course, 2004)

Some years ago a Swedish textbook by Vretblad (1999) was excluded from the list of literature used in the basic university course in order to make room for material aimed at revising upper secondary school mathematics. In Vretblad's book students were introduced to proof and elementary proof techniques in Swedish. For example questions of types (2) and (3) above were introduced and explained in the textbook. Some mathematicians, and some students studying more advanced courses, mentioned the book as a significant help in understanding proof. Also a course in Euclidian geometry was removed from the curriculum at the same time. Hence, students do not practise proving during the basic course at the university as much as earlier. According to the mathematicians, it has not been their practice to discuss proof techniques since Vretblad's textbook was excluded. Thus, the new entrants are trying to understand proof without the systematic guidance of teachers or a textbook.

> But no one has told us that 'Now when you are going to prove statements, consider that…' Rather, you have to try to pick up as much as possible by yourself. (Student – advanced course, 2004)

Students felt that it was expected that they knew what a proof was, and how to construct proofs, from the very beginning of their studies. There were no discussions about proof and many aspects of proof remained invisible for students (Hemmi, 2008).

In summary, Swedish students have various positions concerning their prior proof experiences when they enter university. There is a small minority that has experienced proof in different ways at upper secondary level, whereas many students have neither studied proofs nor practiced proving statements themselves. The students showed an interest in learning more about proof when they started to study mathematics at the university. While mathematicians claimed that they avoided proof in their teaching, and that in general proof has a diminished place in the curriculum, students claimed that the basic course contained a lot of proofs, and they had difficulty following these and in constructing their own proofs.

Development of the Educational System and the Transition Problems

Finally, we address our second question about the possible connections between educational reforms and transition problems. We start this section with an overview of the expansion of higher education in Sweden, focussing on science and engineering, and a description of the development of the transition problems over the last 10-15 years. We describe upper secondary school mathematics reforms, and adjustments made to meet the transition problems.

Mass Education and Universal Access to Higher Education

In most developed countries higher education has gone through a process of growth, which has turned an elite system from the 1950s into a system for mass education in the 2000s. According to Martin Trow (1974) such a process always poses a variety of problems, ranging from funding to recruitment and selection of students. The problems are met in various ways by different countries according to cultural, societal, economical and political factors (Trow, 1979; 1999). The change from elite to mass education with characteristics such as reformed secondary curricula and bridging programmes is an international trend with some common overall features (Hillel, 2001). Swedish reforms at upper secondary and tertiary level (see Tengstrand, 2001 for an overview) during the last decades may be understood as a part of such a process towards mass education and the creation of universal access to higher education.

In spite of a large expansion, too few students went into science and engineering and so universities have been assigned quantified goals by the state to educate more students in these areas since the early 1990s. Many initiatives have been taken in order to attract new groups. The main one—counted by the number of students—is a national bridging programme, installed in 1992, offering students with a background from the Social Science upper secondary programme a one-year programme to get a Natural Science qualification and a guaranteed admittance to a science or engineering programme at university (Sjøberg, 1999).

The efforts to attract students have led to positive results and the proportion of students going into science and engineering increased in Sweden during the 1990s. Sweden, Finland and New Zealand are the three countries among twelve OECD-countries that increased the number of engineering degrees awarded per inhabitant per year from 1992 to 2004 by more than 200%, which is considerably more than the others (National Agency for Higher education, 2007). The number of Swedish new entrants and engineering degrees awarded are shown in Figures 1 and 2 for every fifth year since 1985/86.

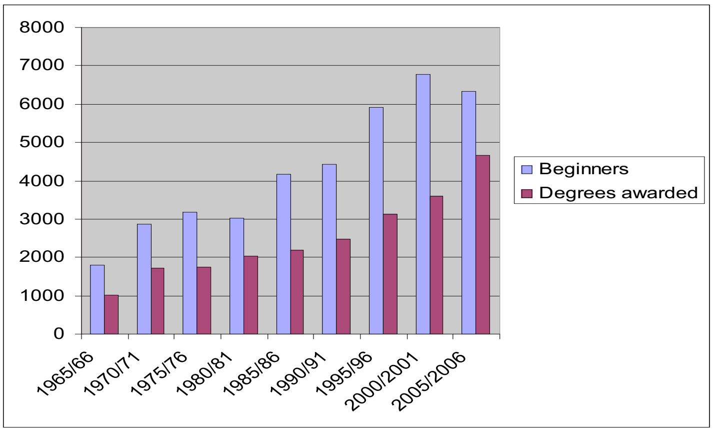

Figure 1. Number of engineering master's degrees and beginning students, certain academic years.

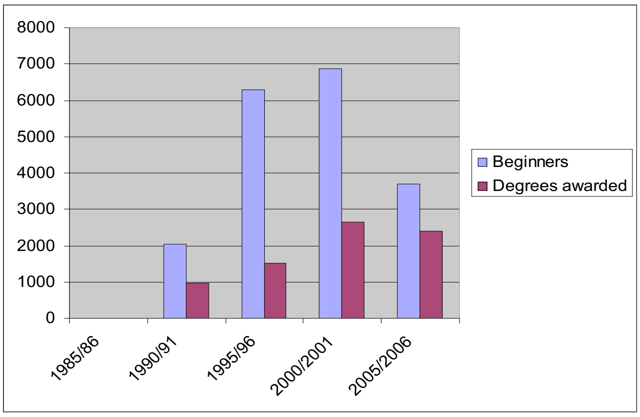

Figure 2. Number of engineering degrees (shorter programme) and beginning students, certain academic years.

Prospective teachers form the second largest group of mathematics students after engineering students, and science students constitute the third largest group. In all these areas transition problems in mathematics are large according to examination results and evaluations.

A Mathematics Education Crisis Occurs During the 1990s

Mathematics performance among students at secondary level and among new entrant university students has been investigated in various studies, admitting comparisons over time. International evaluations of school mathematics such as TIMSS and PISA show a negative trend for Swedish mathematics students

(National Agency for Education 2004a, b), both absolutely and relative to other countries.

Since the beginning of the 1990s, the results from pre-tests given to entering engineering students at several universities have declined. All universities reporting on systematic pre-tests report this trend, to varying degrees. According to a study of entering engineering students there is a strong connection between mathematics grades from school on one hand, and success in mathematics studies at university on the other (Brandell G., 1999). Also, a strong connection between mathematics grades from school and results on the pre-tests is documented in a long-term comprehensive study at KTH (Brandell L., 2007). The transition problems and the declining rate of examination passes appear to be, at least partly, due to inadequate mathematical preparation from school among large groups of students.

Transition problems are not new. Mathematics teachers at universities in Sweden have expressed serious concerns about the working knowledge of mathematics among new entrants since the early 1970s. However, two important factors have most likely increased the transition problems during the last 10-15 years. One is the considerable expansion of higher education in general, described above and the other the school mathematics reforms to which we will return in the discussion.

The significance of the expansion is not completely clear. However, an increasing volume will probably increase the proportion of new entrants with weaker results, which in turn will enlarge the group of students who encounter difficulties. In addition, important new groups of students, with untraditional educational backgrounds, and adult students generally have less stable pre-knowledge in mathematics, according to pre-tests and examination results.

Adjustments at Upper Secondary and Tertiary Level

In order to overcome the transition problems several reforms were implemented in universities, starting in the 1970s. As one example the department at Stockholm University (mentioned in Study 2) has been active in reform, and at the beginning of the 1990s the basic courses in calculus were reformed, as a response to the demands of other practices, like those of natural sciences. A part of the theory, e.g. ε–δ proofs were moved to intermediate courses and instead, more applications and multivariable calculus were included in the basic course in analysis (Hemmi, 2006). Later on, the first course was redesigned to cover introductory topics in order to facilitate the study start. This first course was planned to be largely a revision of upper secondary school mathematics. However, in the light of the results presented in an earlier section (*Curriculum gap and perceptions of mathematics*) it is possible that the contents of the first course are not a revision of upper secondary school mathematics but new material for students.

Since 2000 it is possible for upper secondary schools to offer optional mathematical courses. Many schools use this course as an opportunity to offer a course in mathematics designed specifically for preparing university studies.

The adjustments so far made at tertiary and secondary level to make up for the gap between the levels during the last 10-15 years have not been adequate, or have not been implemented at a sufficient level, since transition problems do not seem to have diminished.

Discussion

Moving from one level to another in the educational system means adapting to a new social context and a new educational system. The students also encounter inherent subject specific difficulties when they move to a more advanced level of studies, such as the reconstruction of mental objects (Artigue, 1999). In this article we have focused on aspects of the secondary-tertiary transition related to curriculum issues and how secondary and tertiary level mathematics fit together in the Swedish educational system.

Main Traits of the Widening Gap

Study 1 shows that there is a considerable mismatch between secondary and tertiary level concerning subject matter as well as competencies. From Study 2 we know that most newcomers at the university have very limited experiences of proofs and deductive reasoning in mathematics. These difficulties are related in several ways. For students who are not confident in their own deductive capacity even on a fairly basic level, formulas and computation algorithms become a hopeless mass of seemingly arbitrary rules to remember.

Proof has many important functions in mathematics (e.g. de Villiers, 1990), and in a broad sense permeates all mathematics as a general way of reasoning and organizing mathematical knowledge in a deductive manner. Hence, working with proofs could improve students' understanding of mathematics in general. Students exercise their algebraic skills, enhance their understanding of general symbols, like the equivalence sign, and learn to organize knowledge when they work with proofs, and the logical structure behind them.

The areas of mathematics where deductive reasoning is most often practised may have fallen into the gap. Hoyles, et al. (2001) point this out and discuss the problem of fragmenting subject matter: "So the gaps in knowledge become not just the lack of bricks, but the gradual disappearance of the cement that holds them together." (p. 15). Put the other way around, understanding and constructing proofs becomes easier for a student whose mind is not preoccupied with uncertainties about formulas, necessary computations and properties of elementary functions.

Since 1994 the curriculum has encouraged the use of computers and calculators in both compulsory school and at upper secondary level. Students use these calculators extensively for calculations and plotting, and calculators are allowed in national tests. Hence, corresponding skills without the help of calculators are emphasized less in the curriculum. However, research shows that students' learning how to use the technology is a process that requires time and careful design of teaching material and instruction (Artigue, 2002). It is far from obvious how teachers may help their students to connect and make use of the different representations of various concepts so easily accessible by both computer and calculator (Kendal & Stacey, 2003). Hence, the introduction of computers and calculators in calculus and algebra courses at secondary level may become an obstacle rather than a means of attaining the goals of procedural and conceptual knowledge. No large-scale competence development programme has been offered to teachers in this domain.

Influence of a Changing System on the Curriculum Gap

An almost universal upper secondary education, broadened access to higher education, visions about life-long learning, recurrent access to higher education and a strong pressure towards expansion of the science-technology sector form the background of the educational reforms at upper secondary and tertiary level in Sweden. This development influences the transition problems, in particular the curriculum gap.

Since the 1994 reform the *structure of upper secondary school* is not well suited for preparing students for further studies in mathematics. The current structure is based on the assumption that mathematics competencies and contents needed for applications in various vocational and theoretical programmes are similar enough to accommodate the same course to the needs of all students. This assumption is not substantiated by research and contradicted by experiences at universities since the reform. Certain competencies, such as giving meaning to algebraic expressions, equations and formulas, and handling of those, have been reduced in school mathematics with reference to the lack of relevance for students other than those in the natural science programme. The same is true for other content, e.g., geometry and functions.

A *gradual shift of emphasis* in the upper secondary curriculum, in terms of subject matter as well as competencies, has taken place without sufficient concern for prerequisites needed for introductory tertiary level. One example is the differing views on the role of technology in mathematics education. Another is the perception of proofs and proving. The upper secondary school mathematics curriculum is not clear about the demands concerning proof, which is one reason behind the varied experiences of the students in Study 2.

Most *universities of technology have lowered the required level* of mathematics for upper secondary school entrants below the national standard. The main reason is their goal to improve recruitment. However, a lowered level of mathematics from upper secondary school may lessen the possibility that students will handle the challenges of new mathematical demands at tertiary level.

It seems clear that at many universities *the first year courses at the tertiary level are overloaded with content*, which creates problems for students. The reason is partly to make up for reductions at upper secondary level and the lower level required for admittance, partly due to expectations from other subjects in science and technology, which require mathematics. However, attempts to adapt the university curriculum to the students' level without any deeper reflections about the general character of mathematical knowledge can lead to overloaded courses with various mathematical topics where students do not see any connections.

In this article we have shed some light from various points of view on the widening gap in Sweden between secondary and tertiary level mathematics education and its influence on transition problems. We have shown that the complexity of the situation is such that no easy solutions can be found and only long-term reforms taking into account both levels can lead to essential improvements.

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