

Fraction Number Line Tasks and the Additivity Concept of Length Measurement⁸

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The performance of 29 Grade 6 children on eight number line tasks is descriptively analysed with regard to three types of misconceptions. The first is conceptual; over-generalised part-whole unit-forming. The other two are semiotic misreadings of inscriptions, namely counting lines and decimalising. Success on these number line tasks, possible examples of the measure sub-construct of fractions, is descriptively compared to the children's performance on tasks assessing conservation and the additivity concept of length measurement, and on multiplication and division items.

Theoretical Framework

The transferability of knowledge within and across domains in mathematics is seen as important to the development of relational understanding. Understanding of fractions leads to proportional reasoning, and fraction understanding is assumed in the study of algebra, probability and geometry (Kieran, 1993). What are the concepts from other domains that support the development of fraction understanding? Links between the domains of fractions and measurement are made from both areas of the research literature. Rational numbers are necessary to describe leftovers that are the result of a non whole number count when measuring (Kieren, 1995). Researchers in the measurement domain also recognise analogous concepts in some measurement and fraction tasks and call for more research on the intersection of the two domains (Lehrer, Jaslow, & Curtis, 2003). The theory of constructivist learning would also suggest cross-domain interaction in understanding, as well as intra-domain trajectories. The connection between performance on integer number lines and measurement tasks has been reported (Pettito, 1990), but the present paper addresses this connection with number lines in the rational number context.

One type of misconception that can occur within a domain, traceable to conceptual over-generalisation, is an error in accounting for the whole during unit-forming. An example of this is in number line tasks, and may appear as finding *half of* a number line with pre-marked partitions rather than where the *number half* goes (Kieren, 1993). This same instrumental part-whole understanding is evident when the child makes their own partitions on number lines, for example, when drawing a number line from 0 to 6 and labelling 4 as $2/3$ (Clarke, Roche, & Mitchell, 2007).

Another family of misconceptions can be attributed to semiotic misreadings of mathematical diagrams or inscriptions. Inscriptions are written representations including both diagrams and symbolic notation, and excluding mental representations (Roth & McGinn, 1998). These misconceptions are a by-product of the use of visual texts to convey information. Mathematical diagrams, both inscriptions and mental visualisations, convey meaning through agreed semiotic conventions on how they are to be decoded (Presmeg, 2006). A behaviour that may link to a semiotic misconception in both the measurement and the fraction domain is counting lines, including the zero-point, and not spaces on scales. In broken ruler tasks, some low ability students revert to counting unit markers rather than linear sub-units (Bragg & Outhred, 2000). Successful students on number line tasks attend to the parts, rather than the vertical lines used to create the equal parts (Bright, Behr, Post, & Wachsmuth, 1988; Pearn & Stephens, 2007). For example, $3/4$ is called $4/5$ if the mark at zero is included in the count. However, strategy use is inconsistent on rational number number lines and similar measurement scales (Drake, 2007). Other problems with nomenclature when identifying fractions on number lines may also have a semiotic component.

The model used in the present research as a criteria for fraction task selection and analysis, is Kieren's sub-construct model which subsumes part-whole relational knowledge into the measure and quotient sub-construct, leaving four sub-constructs- measure, quotient, operator and ratio (1993). The model includes three underpinning concepts- partitioning, equivalence and unit forming. Fractions, decimals and percentages are the notations that can be used to describe mathematical contexts in these sub-constructs. Research on number

8 The authors wish to thank Professor Doug Clarke and Ms Anne Roche for their advice and assistance in the development of the fraction interview protocol and record sheets.

lines in rational number has concentrated on the distinction between making partitions and reading pre-marked partitions, and proper and improper fractions on number lines labelled 0 to 1 and 0 to greater than 1 (Novillis Larson, 1980; Bright, Behr, Post, & Wachsmuth, 1988; Ni, 2000). There has been an assumption in the literature that as a number line can be an example of the measure sub-construct of rational number, that all number lines are measure sub-construct tasks (Ni, 2000). A task is only an example of the measure sub-construct of fractions if that is the reasoning that the child uses when attempting the task. A child may over-generalise from number lines marked 0 to 1 that the ends of the number line form the whole, rather than using the scale to help with unit forming. This can contribute to the first type of misconception described above.

The model for length measurement used as criteria for task selection in this research is based on Lehrer's eight key concepts for spatial measures (2003). We have categorised these eight key concepts into four concepts that need to be co-ordinated; attribute, additivity, units, and proportion. The *attribute* concept includes distinguishing between multiple attributes of a figure, and recognising attributes in increasingly complex formations such as bent paths and perimeters in length measurement. The concept of *additivity* requires an understanding of the zero-point and conservation, that the whole is the sum of the parts including in bent paths. The concept of *units* includes describing both fractional parts of units when count is not a whole number of units, and the inverse relationship between the size of the unit and the count. The concept of *proportion* is important to spatial measures (length, area, volume and angle) involving the recognition that the size of the count is represented proportionally in visual representations of spatial measures.

Number lines are an example of a mathematical context that may enable reasoning from the measure sub-construct of rational numbers and the additivity concept of measurement, because the zero-point is important to both contexts. The inclusion of the semiotic component to the decoding of inscriptions adds another layer to the analysis of children's performance on number line tasks.

Methodology

A one-to one task-based interview, similar in format to the Early Numeracy Interview (Department of Education and Training, 2001), was developed and covered the domains of a) multiplication and division, b) fractions, c) measurement and d) dynamic visualisation or geometric reasoning. The results reported here are preliminary both in terms of sample size and in the narrowing of the scope of this paper to length measurement and fraction number line tasks. The data collection took place in February and March 2008 and included 29 Grade 6 children from two schools in metropolitan Melbourne.

The tasks in the measurement section, analysed in this paper, were chosen to assess relational understanding in the conceptual areas of *additivity*, see *Figure 1*. The interview was piloted in 2006, enabling refinement in all sections and providing evidence that tasks in each of the measurement conceptual sections were hurdle tasks. This enabled the protocol of designating a threshold task for each of the four measurement concepts that every child was asked. Then, if successful they were asked a high benchmark task or if unsuccessful at the threshold task, they were asked the low benchmark task. This divided the cohort, in each of the four conceptual areas, into four sequential groups.

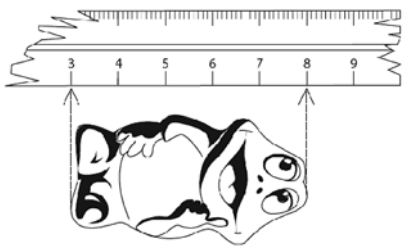
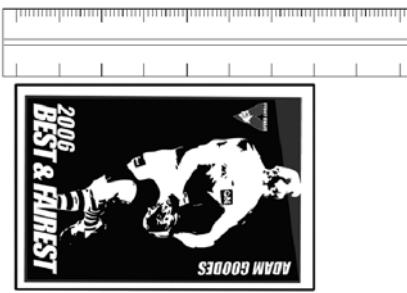
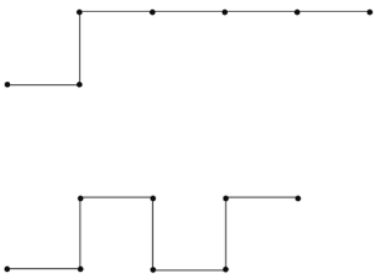
<p>41. Threshold task</p> <p>This centimetre ruler is broken. It is measuring a Freddo frog. How long is the Freddo frog? How did you work that out?</p> <p>Adapted from Bragg and Outhred (2000).</p>	
<p>42. High benchmark task</p> <p>This ruler measures in centimetres but there are no numbers on it. How long is the footy card? How did you work that out?</p> <p>Adapted from Bragg and Outhred (2000).</p>	
<p>43. Low benchmark task</p> <p>These are two pieces of wire that can be bent and straightened. Between the dots is the same length. If the wires were straight, would they be the same length or would one be longer than the other? How did you work that out?</p> <p>Adapted from Battista (2006).</p>	

Figure 1. Additivity conceptual tasks in the measurement interview.

Number line tasks reported in this paper, see *Figure 2*, were one type of task used to assess relational understanding in the measure sub-construct of fractions. Other length and area inscriptions were used for this sub-construct in the interview. Table 1 shows the criteria identified in the research literature that directed the choice of number line tasks. The multiplication and division tasks were taken directly from the Early Numeracy Interview (Department of Education and Training, 2001), but unlike the standard protocol for the early years, after omitting the first two questions, this section of the interview was used in its entirety.


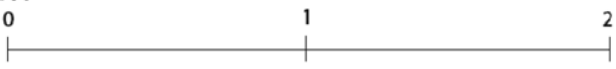
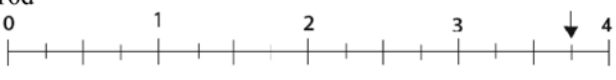
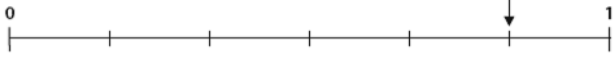

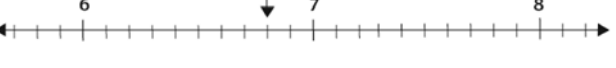
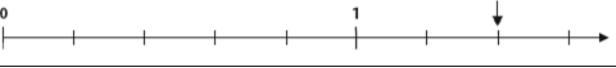
16a. Give the child a blank piece of paper and pen. Please draw a number line and mark two thirds on it. If the child does not mark 0 or 1, ask, where does zero go? Where does one go? How did you work that out? (Clarke, Roche, & Mitchell, 2007)	
If this is half, <i>point</i> , where would one and a half be on this number line? Adapted from Bright, Behr, Post, and Wachsmuth (1988).	16b 
Please mark where one quarter would go on this number line. Adapted from Pearn and Stephens (2007)	16c 
<i>Point to arrow</i> , what number or fraction is that point on the number line? Adapted from Lesh, Landau, and Hamilton (1983).	16d 
<i>Point to arrow</i> , what number or fraction is that point on the number line? Adapted from Novillis Larson (1980).	16e 
<i>Point to arrow</i> , what number or fraction is that point on the number line? Adapted from Pearn and Stephens (2007).	16f 
<i>Point to arrow</i> , what number or fraction is that point on the number line? (Ministry of Education, 2007)	16g 
<i>Point to arrow</i> , what number or fraction is that point on the number line?	16h 

Figure 2. The eight number line tasks used in the data collection interview.

Table 1

The Selection of Number Line Tasks to Represent Research-based Criteria

Task requirements	Number lines 0 to 1	Number lines 0 to >1
Make partitions	16a	Proper fractions: 16c Improper fractions: 16b
Read partitions	16e, 16f (non-equal parts)	Improper Fractions: 16d, 16g, 16h

A ranking system was developed because the sample was quite small, in order to compare children's success on different tasks. A weighted score was calculated for the eight number line tasks based solely on the performance of the cohort and not on outside standards for this age group, even though these were available for some of the tasks. The frequency of success (fs) on each number line task was calculated as a percentage, then, if a child was successful on a task, their score was $100 - fs$, i.e. they performed better than $(100 - fs)\%$ of the students. In this way, they accumulated points for correct answers but more difficult questions were

weighted more highly. An overall number line weighted score was generated. The same protocol was applied to their performance on the tasks from the multiplication and division section of the interview. These different weighted scores enabled the children to be individually ranked from one to twenty-nine on these separate criteria. The ranking process for the additivity tasks was different as the tasks had been chosen because they were sequential. Children's success on the tasks determined to which of four levelled groups they would belong. Each child within a group was equally ranked to the other children within the same group on the additivity tasks.

Results and Discussion

Permutations

There were 25 different patterns of correct and incorrect responses from the 29 children on the eight number line tasks. As four children were successful on all eight tasks and two were not successful on any, there were 21 different patterns of correct and incorrect responses for the 23 children remaining. The pattern of correct responses did not fall into a sequential pattern, although some questions were clearly more difficult (16a) than others (16b). While the children who were successful on four of the eight number line questions all answered 16b correctly, their incorrect responses did not fit a pattern.

Misconception 1: Instrumental part-whole knowledge (difficulties with unit-forming).

Of the 19 children who were unsuccessful at drawing a number line and marking $\frac{2}{3}$, (16a) six constructed a number line using another ratio of $\frac{2}{3}$, for example marking $\frac{2}{3}$ at 6 on a number line marked from 0 to 9. On a separate question, 16c, 11 of the 15 children who placed $\frac{1}{4}$ incorrectly on the number line, marked it at $\frac{1}{2}$ and offered a part-whole explanation that indicated that the whole was the whole number line from 0 to 2. This misconception was not uniformly displayed with only three children using this strategy on both number line tasks 16a and 16c. Conversely, three children demonstrated this unit-forming misconception only on 16a and another eight demonstrated it only on 16c. The frequency of this misconception supports the conjecture made in the theoretical framework section above, that not all number line tasks assess the measure sub-construct of fractions.

Misconception 2: Counting lines not spaces.

There were six children who demonstrated the misconception of counting the lines rather than spaces on one or more of the number line tasks, with one child demonstrating it consistently across four of the tasks. One child, knowing the interviewer could not indicate correct or incorrect answers, verbalised this misconception and rhetorically asked whether he should count the lines or the spaces. This misconception provided the opportunity for two children to get the right answer for the wrong reason (marked incorrect in the data analysis). These two children did not imagine the missing quarter mark on task 16f and counted the zero mark and the two lines marked on the number line. They then called the indicated mark $\frac{3}{4}$ for the wrong reason. Of course, many children counted the vertical lines successfully, either instrumentally knowing not to count the zero or whole number mark, or relationally using the vertical lines as markers of both the end of one unit and the beginning of another. Similarly to Drake's results (2007), demonstration of misconceptions is not consistent, and only one of these six children also counted the lines instead of the spaces on either the Freddo or the footy card task in the additivity section of the measurement interview. There were numerous other errors due to nomenclature issues but they did not indicate the counting of the mark at zero as part of the count, except in some cases of the misconception described below.

Misconception 3: Decimalising the count.

During the coding of responses, a third misconception emerged which also appeared to be a semiotic misreading of the number line inscriptions. The child read the mark as if the whole was divided into tenths. Generally, they counted left to right, so five sixths (16e) became *point five*. One child self-corrected her response of *point five* to *point nine*, because she reinterpreted the number line by reading right to left, and counted back from the whole. The non-equally partitioned $\frac{3}{4}$ (16f.) became *point two* or *point three* depending on whether the counting lines misconception was present or not. Five children demonstrated this misconception but only on one or two questions each. Unsurprisingly, all but one of them gave a correct decimal response to question 16g. This decimalising the count misconception may be an indication of an interference from the

measurement domain, as Australian children use metric measuring instruments and scales may prompt a decimal response. One other child had a specialised decimalising nomenclature system and has not been included in the numbers for the description above. His naming system involved identifying the fractional marking and the whole numbers, for example, $5/6$ became *zero (the first whole number), five (a correct count of the lines) point one (the second whole number)*. On the four tasks that he used this nomenclature, he did not display the counting lines misconception. Equally, other children used decimal notation successfully to describe the points on the number line, most notably in the last two questions.

Transfer between Domains

If the misconceptions that appear similar in measurement tasks and fractions tasks are linked, it is difficult to tell in which direction the transfer might go. It is possible that the transfer might come from either domain, possibly reliant on the classroom activities and mathematical sense making that the child has encountered. It would seem reasonable that there would be a lag been the acquisition of an understanding in one domain, and its transfer to another. If the transfer can happen in either direction, the data may not show a strong correlation.

Success on Number Line Tasks

A comparison between the children’s performance on number line tasks and their additivity grouping showed that this correlation may be worth pursuing with further research, see *Figure 3*. A similar comparison between children’s overall performance on the eight number line tasks, and their performance on the multiplication and division tasks was performed, also shown in *Figure 3*. It is important to remember that the children’s rank order in additivity and in multiplication and division is different. In reading the figure below, the children’s ranking stays consistent down a column but not across the two columns.

Success at the threshold task has been achieved by children 1-20 in the additivity concept grouping, and the eight highest weighted scores for number line tasks are by those children. Conversely, several of the top eight successful children on number line tasks fall in the third quartile of the multiplication and division individual ranking. This showed that the idea of the *successful child* in one domain also being generally successful across several domains was not confirmed by the research. This gives further weighting to the importance of the finding above that success on the number line tasks was correlated with success at the threshold task in the additivity concept tasks.

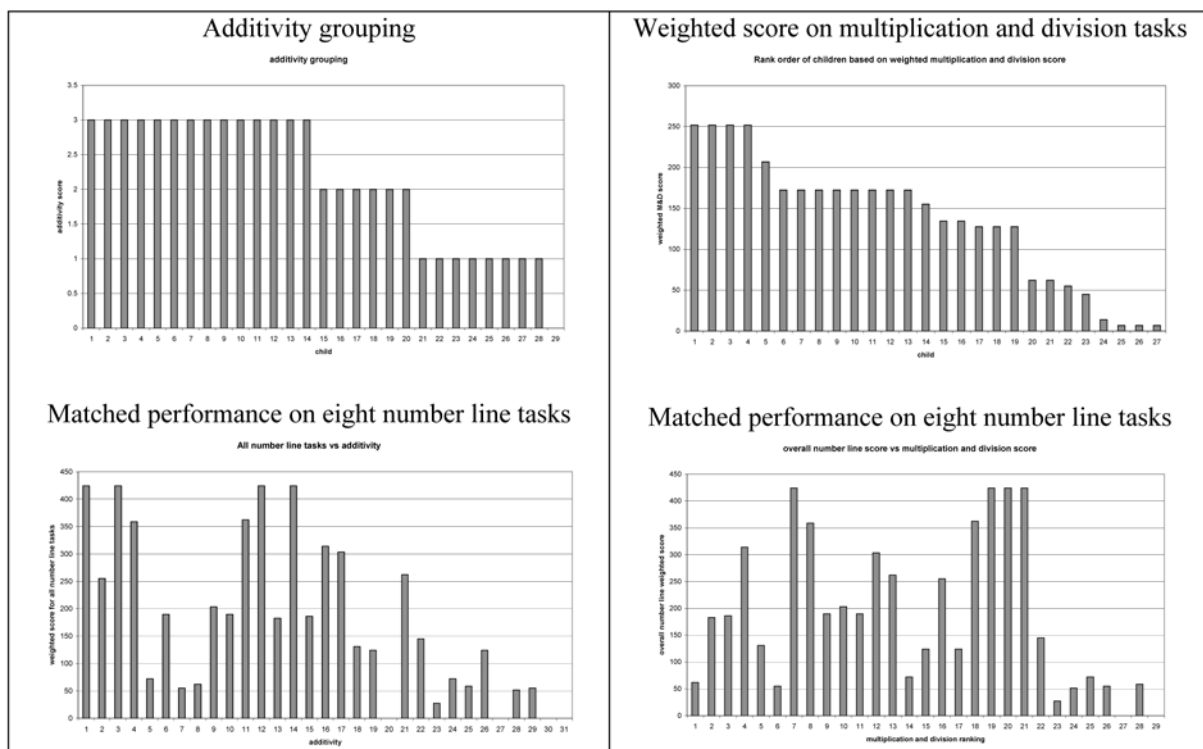


Figure 3. Correlation of number line tasks to additivity and multiplication and division performance.

Conclusions

The variation in patterns of response to the number line tasks highlights the complexity of the task facing teachers in our schools. The range of understandings and strategies amongst our students is broad. Two misconceptions due to semiotic factors were in evidence, counting lines, including the zero-point, instead of spaces, and decimalising the count. A unit-forming misconception related to an instrumental understanding of part-whole knowledge of fractions was also evident in the sample. Success on, as opposed to misconceptions about, number lines correlated more highly with a conceptual understanding of the additivity concept of length measurement than to success at multiplication and division. However, while some children displayed the misconceptions consistently, others only employed them in specific situations, indicating that the counting lines misconception may not be a major contributor to the correlation between additivity grouping and performance on number line tasks. It is unclear whether there is transfer between the two domains of the understanding of the semiotics of lines and spaces. If there is transfer possible in both directions, correlations would be unlikely to be evident. The four conceptual areas of measurement are co-ordinated in an ideal understanding. Number line tasks that use non-equal spacings might draw on the measurement concept of proportion. Number line tasks that use improper fractions might draw on the measurement concept of the unit. Number line tasks in general, because the zero point is crucial to the inscription, may draw on the measurement concept of additivity. Success would seem to be related to a combination of factors rather than the absence of specific misconceptions.

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