Recognising the Difference Between Additive and Multiplicative Thinking in Young Children

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Multiplicative thinking cannot be generalised in any simple way from additive thinking. A premise of this paper is that teachers need to recognise the difference between additive and multiplicative thinking if they are to help children develop the latter. This paper describes an empirical study that investigated the potential of a set of tasks to distinguish between additive and multiplicative thinkers, and illustrates the results through the responses of two children.

Children's prolonged use of additive thinking as a familiar and comfortable means of solving multiplicative problems has been well documented (Resnick & Singer, 1993; Lamon, 1993,1994; Brown, 1981; Fischbein, Deri, & Marino, 1985). Multiplication is more than repeated addition however, and its learning more complicated. While repeated addition may be an appropriate beginning, to maintain that interpretation of multiplication is ultimately disabling because it does not provide children with important multiplicative structures. Multiplicative thinking cannot be generalised in any simple way from additive thinking. Unless teachers consciously help children develop multiplicative thinking, which goes well beyond repeated addition, it may not happen for many children.

This paper reports on part of a larger study (Jacob, 2001) that investigated the development of multiplicative thinking in children. A premise of the study was that unless teachers can actually recognise the difference between additive and multiplicative thinking, it is unlikely they will be able to help children develop the latter. The larger study involved an analytical examination of the research into the mathematics of multiplication and how children learn it. The synthesis of the research provided developmental information about how children move from additive thinking to multiplicative thinking. It suggested five broad phases through which multiplicative thinking develops. These were labelled as *one-to-one counting*, *additive composition*, *many-to-one counting*, *multiplicative relations* and *operating on the operator*. The study also included a small empirical investigation that explored the potential of a particular set of tasks for distinguishing between additive and multiplicative thinkers in a way that could be replicated in classrooms and thus be helpful for teachers. The results of that investigation is the main focus of this paper. The responses to the tasks of two children from the sample will be described, illustrating how these responses would enable teachers to distinguish multiplicative from additive thinking and recognise when children are starting to think multiplicatively. The two questions that will be addressed are:

- What is the difference between multiplicative and additive thinking?
- How can teachers recognise this difference in children?

Research

An analytical examination of the research into the difference between additive and multiplicative thinking requires an analysis of the mathematics of multiplication as well as an analysis of how children develop multiplicative thinking. Research addressing the nature of the multiplication and division operations has tended to take one of three forms. Some researchers have focussed upon essential distinctions between multiplication (and division) problems and addition (and subtraction) problems (Schwartz, 1998; Greer, 1992). Others have studied the characteristics of different types of multiplication and division problems and developed systematic classifications of multiplicative situations (Vergnaud, 1983; Nesher, 1988; Schwartz, 1988; Greer, 1992; Schmidt & Weiser, 1995). A third group of researchers (Davidov, 1992; Boulet, 1998; Clark & Kammi, 1996; Lamon, 1996) has sought to synthesise the above work in order to identify the essential nature of multiplicative thinking that distinguishes it from additive thinking and provides the common element to the various multiplication and division situations. An analysis of the work of the latter group led to the conclusion that it was identification or construction of the multiplicand and the multiplier within a situation and the simultaneous coordination of these factors that typified multiplicative thinking about a situation.

A review of the literature about the conceptual development of children as they learn about multiplication and division included the work of a number of key researchers (Kouba, 1989; Anghileri, 1989; Steffe, 1992; Becker, 1993; Mulligan & Mitchelmore, 1997; Mulligan &Watson, 1998; Battista, 1999) who tended to focus on what children actually say and do rather than on whether or not they succeed on the tasks. It was the work of Kouba (1989), Steffe (1992), Mulligan and Mitchelmore (1997) and Mulligan & Watson (1998) that led to the conclusion that children must first come to recognise multiplicative situations as involving three aspects: groups of equal size (a multiplicand), numbers of groups (the multiplier), and a total amount (the product). When they can construct and coordinate these factors in both multiplication and division problems prior to carrying out the count, they are thinking multiplicatively. Children who think this way would also understand and use commutativity and inverse relations in an empirical way (i.e. by visualising objects). The purpose of the investigation reported in this paper was to find out if these interpretations of multiplicative thinking could be easily recognised when children are solving multiplicative tasks.

Methodology

The empirical study took place in the first term of the year 2000 in a small suburban school in Perth, Western Australia. The school is located in a middle class area with children across the normal achievement range. Fourteen children were selected from a Year Three and a Year Four class. The ages span from seven years and one month to nine years and one month. The children were individually interviewed in a separate room. The interview took place over two sessions of approximately half an hour. The tasks were presented orally to the children and paraphrased if necessary.

Tasks were designed to cover a range of multiplicative situations that could be solved by multiplication or by other methods. The purpose of the interviews was to see how children responded to the tasks and whether their responses could be readily understood and interpreted as using additive or multiplicative thinking. The first set of three tasks involved two- and three-dimensional arrays. The fourth task involved the relationship between multiplication and division while Tasks 5, 6 and 7 presented multiplicative situations that could not immediately be thought of as repeated addition. The final task (Task 8) involved grouping and counting quantities in ways that do not change the quantity, and commutativity. (See Appendix I for a description of the eight tasks.)

Data collected from the individual interviews were analysed and synthesised so that it was possible to, firstly, examine how the children were thinking in terms of the first four developmental phases identified above and secondly, recognise the difference between additive and multiplicative thinking. As previously stated the focus of this paper is on the second set of analyses.

Results

In order to illuminate the difference between an additive and a multiplicative thinker the differences between what Ruby (8 years 2 months) and Nathan (9 years 1 month) said and did will be described for a number of the tasks.

The Array Structure tasks

In this cluster of tasks children were asked to say how many squares and blocks in two and three-dimensional arrays (Tasks 1 and 2). It also includes the Picnic Blanket Squares task (Task 3) which presents an array in the form of a picnic blanket that was partially covered by trays of food.

*Ruby***.** Ruby used the equal grouping structure of the row and columns of a prism made from 2 cm cubes (Block Towers) in order to skip count. For example, when asked to count the 3 x 3 x 2 array Ruby's response was "OK. 6, 12, uh, 6, 12, 18." Ruby looked for the equal groups (which were easily seen) and used the number in each group to skip count. Her focus was on the number in each group. This strategy did not require her to construct a multiplier from the situation.

In the Picnic Blanket Squares task, however, the groups were not as easy to recognise. For Ruby to use the same skip counting strategy she would first have had to construct the groups. Ruby did not construct groups in that array even though constructing a group of five would have enabled a far easier count than the count by ones she carried out. The following excerpt is of Ruby carrying out the Picnic Blanket Squares task

Int: I want to know how many squares on the picnic blanket. The kids were actually standing around the edge trying to work out how many squares on the picnic blanket.
Ruby: So even those.
Int: Even the hidden ones.
Ruby: (She had counted the visible squares in the first two rows and part of the third row) OK. 13, 14, 15, 16, 17, 18, 19, 20.

Nathan. In contrast, Nathan constructed a multiplicand and a multiplier in the Picnic Blanket Squares task and used them to multiply. Not only that, but he went one step further and used the commutative property of multiplication.

Int: Can you help them work out how many squares on a picnic blanket?
Nathan: 20.
Int: 20. So how did you work that out? What did you say to work that out?
Nathan: Um. I counted the top rows so, I counted the top rows and there is five up there and then I counted down the bottom [indicating down the side] and there is four.
Int: Mm.
Nathan: And then I turned, I did the turn about and I switched it around and it was four times five. And four times five is twenty.

The Multiplication/Division Relationships Tasks

The children were first asked how many piles of three blocks they could get from twelve blocks. Immediately after they had answered they were asked how many blocks are in four piles of three (Task 4).

*Ruby***.** Ruby used addition to think about both questions. She used repeated addition for the division question, counting the groups in her head. However, she did seem to know the answer to the multiplication question without calculating although she explained it through repeated addition. Ruby solved the Block Piles task in the following way:

Int: How many piles of 3 could you get from 12 blocks?
Ruby: Um, 3 ... 4.
Int: And how did you know that?
Ruby: Because um 9. I didn't know how to equal 9 by equalling 3. 3 plus 3 equals 6, another 3 equals, and another 3 equals 12.
Int: So how many piles of 3 could you get from 12 blocks?
Ruby: 4
Int: How many blocks would you have in 4 piles of three?
Ruby: Four piles of blocks would be 12, um, something near to I'd say. Oh 12.
Int: OK how did you work that out?
Ruby: Just the same as the other one. 3 pus 3 is 6 and another 3 plus 3 equals 12.

Nathan. Nathan used multiplication, multiplication in parts and the inverse relationship to answer the questions. He carried out a multiplication in parts to solve the division. Once he had solved the division he established the multiplier, the multiplicand and the product and was able to answer the multiplication question using that information.

Nathan: Four.
Int: Right that was quick again. How did you work out it was four?
Nathan: Well, you know how 3 times 3 is 9?
Int: Yes
Nathan: Well I took away 3 and then, and then I added 3 more onto 9 and I had 12
Int: How many blocks in 4 piles of 3
Nathan: How many blocks in 4 piles of 3? Uh! 12.
Int: How did you know that?
Nathan: Hhh. Because 4 times 3 equals 12.

The Multiplicative Situations Tasks

In an equal group problem such as "There are 4 baskets with 3 apples in each basket. How many altogether?" the multiplicand and the multiplier are already made explicit for the child. This is not the case with problems that are not of the equal group kind. The children were asked to solve a simple conversion (Task 5), multiplicative compare (Task 6), and a Cartesian product (Task 7) problem. In order to solve the problems using a strategy that was not a straight count, the multiplicand and the multiplier has to be constructed from the situations. Only responses to the simple conversion problem (Pattern Block Task 5) will be reported here.

Ruby. In the Pattern Blocks task, Ruby was able to solve the problems by using the multiplicand as can be seen by the way she used skip counting in her explanations. The pattern block task was a straight skip count that continued until she had counted three in each of the imaginary hexagon shapes. She did not construct and did not need to construct the multiplier using this strategy.

Int: How many of these blue shapes do you think would fit in the yellow shapes.
Ruby: Three.
Int: OK. So how many blue shapes do you think would fit in there?
Ruby: Um. 3, 6, 9, 12, 15. No. 9, 12. That's it yep. Fifteen.

Nathan. Nathan was able to construct a multiplicand and a multiplier and coordinate them in the multiplicative problems where the groups were not at all obvious.

Nathan: There would fit, fifteen.
Int: How did you think about that?
Nathan: Three. I put, I still, I thought of those still on there, on there, and I thought it more carefully and there were, and, there were three on each, there was five times three, I didn't actually get the answer to that one but I turned it around, I did a turn around and it was three times five and that equals fifteen.
Int: Because you know three fives better do you.
Nathan: Yep.

In this construction, the multiplicand was not the focus in the same way it was for Ruby. For Nathan, the multiplicand almost becomes subordinate. He operates through the multiplier as can be seen from the way he says "five times three". Not only that but he operated further and used the commutative property to produce an easier calculation.

The Different Groupings and Commutativity Task

For Task 8, the children were first asked to imagine making a pile of playing cards from one to ten for each of the suits (that is, the hearts, spades diamonds and clubs). They were then asked, "How many cards altogether? How many piles would there be? How many cards in each pile?" They were then asked to imagine making different piles with the cards, this time with all the aces in one pile, all the twos in another pile, all the threes in another, and so on. They were again asked, "How many cards altogether? How many piles would there be? How many cards in each pile?"

Children who have generalised the idea that it doesn't matter how you count a collection the quantity remains the same, are confident that the number of cards doesn't change with the different arrangement. A fully additive thinker would understand this idea.

*Ruby***.** Ruby needed to calculate the number of cards in the second arrangement.

*Nathan***.** Nathan was confident that the quantity of cards did not change with the different arrangement. He also demonstrated that he understood commutativity in an empirical way. That is he could imagine the rearrangement of the cards and see in his minds eye that four piles of ten could be rearranged to make ten piles of four.

After arranging the cards differently Nathan was asked, "How many cards?"

Nathan: We'd still have forty.
Int: Oh. We'd still have forty. Why is that?
Nathan: Oh. Well. One for the ... [indecipherable] you didn't take out any cards except the picture ones and then you didn't take any cards out for the next one so there was forty.
Int: You're right. So how many cards in each pile would you have?
Nathan: Four.
Int: And how many piles would you have?
Nathan: You would have ten.

Conclusion and Implications

It is possible to see from these two children's responses what it means to start to think multiplicatively and how this thinking can be distinguished from additive thinking. Ruby operated in ways that distinguished her as an additive thinker. Nathan operated in ways that distinguished him as a multiplicative thinker. The difference between their thinking has been made explicit through an examination of their responses to a range of multiplicative tasks.

Ruby recognised the grouping structure in some of the multiplicative tasks but not all of them. In situations where she could recognise a grouping structure she focused on the number in each group and this enabled her to skip count or repeatedly add to solve the problem. Where she did not recognise or could not construct the number in each group and the number of groups she resorted to one-to-one counting. Nathan was able to recognise grouping structures in the tasks where the number of groups and the number in the groups were not obvious. He was able to construct a multiplicand and a multiplier out of these situations and coordinate them as a multiplication. Also it was possible to see that, for Nathan, the number in each group became subordinate to the number of groups. He tended to focus on the multiplier in the situations not the multiplicand. Nathan not only used multiplication for multiplication problems but also understood how multiplication can be used to solve division problems. He showed his understanding of commutativity in an empirical way in the cards task but also used it independently a number of times in order to provide an easier calculation with which to solve the tasks.

Nathan has demonstrated through his responses that he is thinking multiplicatively, however, he still has a way to go before he could be considered to be a fully multiplicative thinker. A fully multiplicative thinker would be able to *operate on the operators*. Nathan's focus on the multiplier in his operations, which is a signal to his thinking, needs to be extended in a number of ways. He needs to be able to add and subtract groups using part whole understandings in both multiplication and division situations. Further to this, he would need to understand how groups themselves, or the multiplier, can itself be multiplied and divided. This understanding would enable him to solve a more complex type of multiplicative problem such as described by Schmidt and Weiser (1995) as 'The structure of composition of operators.' They give an example of this type of problem: *During his first year of life Otto the elephant trebles his weight at birth in his second year he doubles his weight. What multiple of his weight has he got at the end of his second year of life?* (p. 60)

A major aim of this study was to provide tasks that would enable teachers to recognise multiplicative thinking when it occurs. The first researcher, who is a primary teacher, was able to do this with a range of tasks that are not unlike the types of activities that occur in the classroom and within a modest amount of time. However this was not an easy task and we cannot assume that teachers could take these tasks and come to the same conclusion within the normal classroom context. Further developmental work would be necessary before this information can be made accessible to teachers. This may take the form of working with a small group of teachers who, at first, use the tasks in the controlled way used in this study in order to see if they get the same types of responses. It would then be necessary to see if, with support, they could interpret the responses and hence distinguish multiplicative thinkers from additive thinkers.

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Appendix: Description of Tasks

Task 1: Arrays. Arrays of two centimetre squares were glued onto card and revealed one at a time to children. They were asked: *How many squares?* and *How did you work it out?* The arrays were 3 x 4, 2 x 6 and 4 x 3.

Task 2: Block Towers. Two different three-dimensional prisms, constructed from two centimetre wooden cubes, were presented to the children. The first block tower was 2 x 2 x 4 cubes, the second 2 x 3 x 3 cubes. The children could pick up the towers and examine them. They were asked: *How many blocks in the tower?* and *How did you work it out?*

Task 3: Picnic Blanket Squares. Children were told that a large plate of choc chip cookies and a tray of drinks were sitting on a picnic blanket, which had a square pattern on it. (See Figure 2.) They were asked if they could help the children at the picnic work out how many squares were on the blanket without shifting the food and drinks. They were then asked to explain how they worked it out.

Task 4: Block Piles. A pile of twelve blocks was placed in front of each child to see but not handle. The child was then asked: *How many piles of three could you make from the blocks?* Immediately children answered, they were asked: *How many blocks would be in four piles of three?*

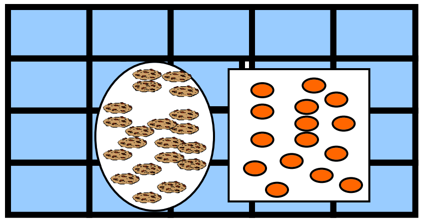

Figure 2. Diagram used for Task 3.

Task 5: Pattern Blocks. The child was shown one hexagon pattern block, one red trapezium and one blue rhombus and an outline of a shape that had been produced by tracing around five pattern blocks placed together. The child was shown the latter shape and asked a series of questions: *How many yellow hexagons like this one does it take to cover this shape? Do you want to check using this one? How many red trapezoids does it take to fit on a hexagon? So how many do you think will fit on the shape drawn here? How did you work it out?* Following this, children were asked: *How many blue rhombuses does it take to fit on a hexagon? So how many do you think will fit on the shape drawn here? How did you work it out?*

Task 6: Fish Pellets. The child was shown two yellow fish the same shape as each other but with one much bigger and two blue fish also the same shape as each other but with one much bigger. They were first asked: *This fish* (pointing to the small yellow fish) *needs 3 pellets of food, but this fish* (pointing the large yellow fish) *is much bigger and needs 4 times that amount. How many pellets should we feed to this fish* (pointing to large yellow fish)*?* They were then asked: *This large blue fish* (pointing to the fish) *needs nine pellets of food. This is three times as much as this little fish needs* (pointing to the small fish). *How many pellets should we feed to the little fish?* The fish were not made available for the children to handle.

Task 7: Outfits. Cardboard coloured cut outs of a set of shorts and a set of shirts were shown to the children and the following scenario was presented to them: *A child has five different coloured shirts* (showing the different shirts). *S/he also has three different colours of shorts* (showing the different shorts). *By changing which T-shirts s/he wears with which shorts s/he could have different outfits couldn't s/he? For example, one day s/he could be all dressed in red. The next day s/he could have on the brown shorts with the yellow shirt. That would be a different outfit wouldn't it? And if he changed them around — tried each pair of shorts with each of the different T-shirts — how many different ways could s/he look? How many different outfits could s/he make from these?* The cut out shapes were not available for the children to handle.

Task 8: Cards. It was explained that the picture cards were removed for this task. The children were first asked to imagine making a pile of cards from one to ten for each of the suits (that is, the hearts, spades diamonds and clubs). The interviewer demonstrated what she meant by putting some of the hearts out in order from one to five. They were then asked: *How many cards altogether? How many piles would there be? How many cards in each pile?* The children were then asked to imagine making different piles with the cards, this time with all the aces in one pile, all the twos in another pile, all the threes in another, and so on. They were again asked: *How many cards altogether? How many piles would there be? How many cards in each pile?*